

37. Appendix: the derivation of equations determining the commodity composition of output from joint-product industries, discussed in subsection 20.2

This appendix sets out the theoretical underpinnings of E_{x0com} , E_{x0ci} and E_{p0com} specifying commodity outputs from joint-product industries.

We consider a firm which chooses its outputs of commodities X_i , $i \in COM$ to

$$\text{maximize }^{56} \quad \sum_i P_i X_i \quad (37.1)$$

$$\text{subject to} \quad \sum_{c \in CC} (Q_c A_c / Z)^{h_c} / h_c = \beta \quad (37.2)$$

and

$$Q_c = \left[\sum_{i \in M(c)} (X_i / B_i)^{-\rho_c} \right]^{-1/\rho_c} \quad \text{for all } c \in CC, \quad (37.3)$$

where

P_i is the price of commodity i ;

CC is the set of composite commodities (denoted by $COMPCOM$ in section 18);

Q_c is the output of composite commodity c ;

Z is an index representing the firm's level of inputs;

A_c is a variable allowing for composite-commodity- c expanding technical change;

h_c , $-\rho_c$ and β are positive parameters with h_c and $-\rho_c > 1$;

$M(c)$ is the set of MONASH commodities that make up composite commodity c ; and

B_i is a positive variable allowing for commodity- i expanding technical change.

Expression (37.1) is the firm's revenue while (37.2) and (37.3) specify its transformation frontier. Equation (37.2) is the CRETH form introduced by Vincent *et al.* (1980). It is analogous to Hanoch's (1971) CRESH specification of the production function. Just as CRESH generalizes the CES production function, CRETH generalizes the CET transformation frontier. Equation (37.2) reduces to the CET case (Powell and Gruen, 1967 and 1968) if $h_c = h$ for all c .

In (37.3) we assume that each composite commodity, c , is a CET function of MONASH commodities, i . Because each commodity i is in only one composite commodity, X_i appears in just one of the constraints that make up (37.3).

The first-order conditions for a solution of (37.1)-(37.3) can be expressed as:

⁵⁶ For algebraic convenience we assume that the firm can produce all of the commodities in COM .

$$\Lambda Q_c^{h_c-1} (A_c/Z)^{h_c} = \Pi_c, \quad c \in CC, \quad (37.4)$$

$$P_i = \Pi_c \left(\frac{X_i^{-\rho_c-1}}{B_i^{-\rho_c}} \right) Q_c^{1+\rho_c}, \quad i \in \text{COM and } c \in CC: i \in M(c) \quad (37.5)$$

plus the constraints (37.2) and (37.3)

where Λ and Π_c are the Lagrangian multipliers for (37.2) and (37.3). Notice that (37.5) and (37.3) imply that Π_c is the price of composite commodity c , i.e. it is the revenue derived by the firm per unit of output of composite commodity c .

In percentage-change form (37.4), (37.2), (37.3) and (37.5) are:

$$\lambda + (h_c - 1)q_c - h_c(z - a_c) = \pi_c, \quad c \in CC, \quad (37.6)$$

$$\sum_{c \in CC} R_c q_c = z - \sum_{c \in CC} R_c a_c, \quad ^{57} \quad (37.7)$$

$$q_c = \sum_{i \in M(c)} (x_i - b_i) H_i \quad (37.8)$$

and

$$p_i = \pi_c - (\rho_c + 1)x_i + \rho_c b_i + (1 + \rho_c)q_c, \quad i \in \text{COM and } c \in CC: i \in M(c) \quad (37.9)$$

where the lowercase λ , q_c , p_i , z , a_c , π_c , x_i and b_i are percentage changes in the variables denoted by the corresponding uppercase letters, and the R_c are shares (summing to one) defined by

$$R_c = \frac{(Q_c A_c / Z)^{h_c}}{\sum_{s \in CC} (Q_s A_s / Z)^{h_s}}, \quad c \in CC, \quad (37.10)$$

and the H_i are shares [summing to one over the i 's in $M(c)$ for any $c \in CC$] defined by

$$H_i = \frac{(X_i / B_i)^{-\rho_c}}{\sum_{k \in M(c)} (X_k / B_k)^{-\rho_c}}, \quad i \in \text{COM and } c \in CC: i \in M(c). \quad (37.11)$$

⁵⁷ In percentage-change form (37.2) can be written as

$$\sum_{c \in CC} (q_c + a_c - z) \bar{R}_c = 0, \quad (a)$$

where $\bar{R}_c = (Q_c A_c / Z)^{h_c}$. If the \bar{R}_c s are converted to shares summing to one then (a) reduces to (37.7).

By multiplying both sides of equation (37.4) by Q_c and substituting into (37.10) we find that R_c is the share of composite commodity c in the firm's total revenue. By multiplying both sides of equation (37.5) by X_i and substituting into (37.11) we find that H_i is the share of commodity i in the value of the firm's output of the composite commodity c to which i belongs.

Substituting (37.8) into (37.9), then multiplying by H_i and summing over $i \in M(c)$ we obtain

$$\pi_c = \sum_{i \in M(c)} p_i H_i + \sum_{i \in M(c)} b_i H_i, \quad (37.12)$$

which can be used to eliminate π_c from (37.9) giving:

$$x_i = q_c + \sigma_c \left[p_i - \sum_{i \in M(c)} p_i H_i \right] + b_i + \sigma_c \left[b_i - \sum_{i \in M(c)} b_i H_i \right] \quad i \in \text{COM and } c \in \text{CC}; i \in M(c), \quad (37.13)$$

where σ_c is a positive parameter defined by

$$\sigma_c = \frac{-1}{1 + \rho_c}.$$

Now we manipulate (37.6) and (37.7) to obtain an expression for λ which is independent of q_c , $c \in \text{CC}$:

$$\lambda = z - \sum_{s \in \text{CC}} R_s^* a_s + \sum_{s \in \text{CC}} R_s^* \pi_s, \quad (37.14)$$

where R_s^* is the modified revenue share defined by

$$R_s^* = \frac{R_s / (h_s - 1)}{\sum_{c \in \text{CC}} R_c / (h_c - 1)}, \quad s \in \text{CC}. \quad (37.15)$$

On substituting from (37.14) into (37.6) and making some rearrangements, we obtain

$$q_c = z + \tau_c (\pi_c - \sum_{s \in \text{CC}} R_s^* \pi_s) - a_c - \tau_c (a_c - \sum_{s \in \text{CC}} R_s^* a_s), \quad c \in \text{CC}, \quad (37.16)$$

where τ_c is the positive parameter defined by

$$\tau_c = 1 / (h_c - 1), \quad c \in \text{CC}. \quad (37.17)$$

To go from (37.16), (37.13) and (37.12) to E_{x0ccom} , E_{x0ci} and E_{p0ccom} requires three steps. First we add industry identifiers, j . Second, we allow the technological change variable a_c to be the sum of two parts, one depending only on j [$a_{0ind}(j)$] and the other depending on both c and j [$a_{0ccom}(c,j)$]. Similarly, we allow the technological change variable b_i to be the sum of two parts, one depending

only on i [$a_{0com}(i)$] and the other depending on both i and j [$a_{0ci}(i,j)$]. Notice in E_{x0ccom} that the $a_{0ind}(j)$ does not appear in the technology-substitution term, i.e., the last term on the RHS. Non-zero values for $a_{0ind}(j)$ do not have differential impacts the firm's ability to produce different composites and therefore do not induce transformation between composites.

The third step in moving from (37.16), (37.13) and (37.12) to E_{x0ccom} , E_{x0ci} and E_{p0ccom} involves changes in the names of variables, coefficients and parameters: q becomes x_{0ccom} ; z stays as z ; a becomes a_{0ind} plus a_{0ccom} ; τ becomes $SIG0CC$; σ becomes $SIGMACC$; π becomes p_{0ccom} ; p becomes p_{0dom} ; x becomes x_{0ci} ; $-b$ becomes a_{0com} plus a_{0ci} ; R^* becomes $MH0CC$; and H becomes $S0C$. The coefficient $CCPROD(i,j,cc)$ is used in E_{x0ci} and E_{p0ccom} to handle the restriction of sums to values of i in $M(c)$ and to identify the c in CC such that $i \in M(c)$.

38. Appendix: the derivation of the equation (E_{x1csi}) determining the domestic/import mix in intermediate inputs, discussed in subsection 20.3

Starting from a cost-minimizing description of domestic/import choice, we derive the demand equation for intermediate inputs, E_{x1csi} , and demonstrate that the "a", "twist" and "f1" terms on the RHS of this equation can be interpreted as input-using technical changes.⁵⁸

We assume that firms in industry j choose inputs of domestically produced (dom) and imported (imp) good i to minimize the costs of satisfying their requirements for i . Omitting i and j subscripts, we write their cost minimizing problem as:

$$\text{choose } X_s, s \in \text{SOURCE to minimize } \sum_s P_s X_s \quad (38.1)$$

$$\text{subject to } R = \left[\sum_s (X_s / B_s)^{-\rho} \right]^{-1/\rho}, \quad (38.2)$$

where

SOURCE is the set {dom, imp};

P_s is the price to industry j of input i from source s ;

R is a CES index representing the required overall level of input i to industry j ;

ρ is a parameter whose value is greater than -1 , but not precisely zero⁵⁹; and

B_s is a variable which allows for commodity-(i,s)-using technical change in industry j .

We look for a specification of the movements in B_s which, in combination with (38.1) and (38.2), gives E_{x1csi} .

With the Lagrangian multiplier denoted by Λ , the first-order conditions for problem (38.1)-(38.2) can be written as

$$P_t = \Lambda \left[\sum_s (X_s / B_s)^{-\rho} \right]^{-(1+\rho)/\rho} X_t^{-\rho-1} / B_t^{-\rho}, t \in \text{SOURCE}, \quad (38.3)$$

plus the constraint (38.2).

By using (38.2), we can rewrite (38.3) as

$$P_t = \Lambda R^{1+\rho} X_t^{-\rho-1} / B_t^{-\rho}, t \in \text{SOURCE}. \quad (38.4)$$

In percentage change form (38.4) and (38.2) are:

$$p_t = \lambda + (1+\rho)r - (1+\rho)x_t + \rho b_t, t \in \text{SOURCE}, \quad (38.5)$$

and

⁵⁸ Throughout this section we allow input-using technical changes of both signs. Input-saving technical changes are negative input-using technical changes.

⁵⁹ As ρ approaches zero, (38.2) approaches the Cobb-Douglas function. In the interests of saving space, we will not deal with this case explicitly. Textbook treatments include Dixon *et al.* (1980), pp. 295-300.

$$r = \sum_s S_s (x_s - b_s) , \quad (38.6)$$

where

$$S_s = (X_s / B_s)^{-\rho} / \sum_t (X_t / B_t)^{-\rho} . \quad (38.7)$$

From (38.4), we see that

$$S_s = P_s X_s / \sum_t P_t X_t, \quad s \in \text{SOURCE}, \quad (38.8)$$

i.e., S_{dom} and S_{imp} are domestic and import shares in industry j 's expenditure on i .

On multiplying (38.5) through by S_t and summing, we find that

$$\lambda = \sum_t S_t p_t - (1 + \rho)r + (1 + \rho) \sum_t S_t (x_t - b_t) + \sum_t S_t b_t . \quad (38.9)$$

Simplifying via (38.6) gives

$$\lambda = \sum_t S_t p_t + \sum_t S_t b_t . \quad (38.10)$$

By substituting from (38.10) into (38.5) we obtain

$$x_s = r - \sigma(p_s - \sum_t S_t p_t) + b_s - \sigma(b_s - \sum_t S_t b_t), \quad s \in \text{SOURCE}, \quad (38.11)$$

where σ is the elasticity of substitution between the two sources of input i , given by

$$\sigma = 1/(1 + \rho) . \quad (38.12)$$

How can we specify the b_s 's so that (38.11) becomes $E_{x1}c_{si}$? We start by writing b_s as the sum of two parts:

$$b_s = a_s + C(s)\text{twist}, \quad s \in \text{SOURCE} \quad (38.13)$$

where a_s and twist are variables and $C(s)$ is a coefficient. On substituting from (38.13) into (38.11), we obtain

$$x_s = r - \sigma(p_s - \sum_t S_t p_t) + a_s - \sigma(a_s - \sum_t S_t a_t) + [C(s) - \sigma(C(s) - \sum_t S_t C(t))]\text{twist}, \\ s \in \text{SOURCE} . \quad (38.14)$$

If we are to move from (38.14) to $E_{x1}c_{si}$, then $C(s)$ must satisfy

$$C(s) - \sigma(C(s) - \sum_t S_t C(t)) = -(\text{SOURCE}_{\text{DOM}}(s) - S_{\text{dom}}), \quad s \in \text{SOURCE}. \quad (38.15)$$

Because $\rho \neq 0$, σ is not equal to one, allowing us to satisfy (38.15) with⁶⁰

$$C(s) = -(\text{SOURCE}_{\text{DOM}}(s) - S_{\text{dom}})/(1 - \sigma), \quad s \in \text{SOURCE}. \quad (38.16)$$

Now, with the input-using technical changes (b_s) written as

⁶⁰ If σ were one, then it is clear that (38.15) would not generate values for $C(s)$. This failure reflects the fact that with a Cobb-Douglas specification ($\sigma=1$) of production technology, input-using technical change does not twist the cost-minimizing domestic/import mix.

$$b_s = a_s - \left[\frac{(\text{SOURCEDOM}(s) - S_{\text{dom}})}{(1 - \sigma)} \right] \text{twist}, s \in \text{SOURCE}, \quad (38.17)$$

(38.11) becomes

$$x_s = r - \sigma(p_s - \sum_t S_t p_t) + a_s - \sigma(a_s - \sum_t S_t a_t) - (\text{SOURCEDOM}(s) - S_{\text{dom}}) \text{twist}, \\ s \in \text{SOURCE} . \quad (38.18)$$

From (38.18) we can move to E_x1csi in four steps. First, we add i and j subscripts to all the variables, coefficients and parameters with the following exceptions. The parameter $\text{SOURCEDOM}(s)$ needs neither i nor j . The parameter, σ , needs only an i because MONASH does not contain different values for domestic/import substitution elasticities for different industrial users of good i . Similarly, the variable, twist , needs only an i . MONASH allows only a common domestic/import twist across all users of i .

The next step is to recognize that under the Leontief assumption adopted at level 1 (Figure 20.1) for industry input technologies,

$$r_{ij} = z(j) + a1(j) + a1ci(i, j), \quad (38.19)$$

i.e., the percentage change in the overall requirement for input i in industry j is the percentage change in the industry's overall level of activity [$z(j)$], modified by general input-using technical changes [$a1(j)$] and input- i -using technical changes [$a1ci(i, j)$].

After substituting (38.19) into (38.18), the third step in the journey from (38.18) to E_x1csi is to rename variables, coefficients and parameters: x becomes $x1csi$; σ becomes SIGMA1 ; p becomes $p1csi$; S becomes SOURCE_SHR1 ; a becomes $a1csi$; SOURCEDOM stays unchanged; and twist becomes twist_src . At this stage, we have arrived at E_x1csi , except for the "f1" terms.

The final step is to add these terms. As explained in subsection 20.3, they occur with non-zero coefficients only in the demand equations for imported transport services (C99) and imported communications (C100). For both transport services and communications, $\text{SIGMA1}(i)$ is zero. This allows us to think of $f1_trans(j)$ as an additional imported-C99-using technical change [having the same effect as $a1csi(\text{C99}, \text{imp}, j)$] and $f1_commun(j)$ as an additional imported-C100-using technical change [having the same effect as $a1csi(\text{C100}, \text{imp}, j)$].

With one insignificant exception, we conclude that E_x1csi is consistent with a cost-minimizing description of domestic/import choice in which technical change is of the input-using variety. The exception occurs in the Cobb-Douglas case (σ is precisely one). In this case the twist term in E_x1csi cannot be interpreted as input-using technical change. In any other case (σ not precisely one) we see from (38.17) that a value of one for $\text{twist_src}(i)$ imparts input- (i, dom) -using technical change of $-S_{\text{imp}}/(1 - \sigma)$ per cent and input- (i, imp) -using technical change of $S_{\text{dom}}/(1 - \sigma)$ per cent.

39. Appendix: the derivation of the equations in subsection 18.8c determining industry demands for primary factors, discussed in subsection 20.4

Our aim in this appendix is to set out the theoretical underpinnings of E_{labind} , $E_{cap_at_t}$, E_{p1land} , $E_{x1laboi}$ and E_{p1lab} . We show that they are the outcome of a cost minimization problem in an environment of input-using technical change.⁶¹

We consider a firm which chooses inputs of labour (X_1), capital (X_2) and land (X_3), and of occupational hours (X_{1q} , $q=1, \dots, M$) to minimize

$$\sum_{m=1}^M P_{1m} X_{1m} + P_2 X_2 + P_3 X_3 \quad (39.1)$$

subject to
$$\sum_{c=1}^3 (X_c / Z B_c)^{h_c} / h_c = \beta \quad (39.2)$$

and

$$\sum_{m=1}^M (X_{1m} / X_1 B_{1m})^{k_m} / k_m = \gamma \quad (39.3)$$

where

P_{1m} is the cost per unit of labour of occupation m ;

P_2 and P_3 are the user prices of capital and land;

Z is the firm's overall activity level;

B_c is a positive variable allowing for input- c -using technical change;

B_{1m} is a positive variable allowing for occupation- m -using technical change;
and

β , γ , h_c and k_m are parameters with h_c and k_m less than 1 but not precisely zero.⁶²

The minimand, (39.1), is primary factor costs. The first constraint, (39.2), means that the firm must obtain sufficient primary factor inputs to achieve the activity level Z where Z is a CRESH aggregate of inputs of labour, capital and land. The second constraint, (39.3), means that the firm must obtain sufficient occupational inputs to achieve an overall labour input of X_1 where X_1 is a CRESH aggregate of occupational inputs.

Problem (39.1)-(39.3) can be solved in two steps. In the first step, the firm chooses its occupational inputs (X_{1m}) to minimize

⁶¹ As in the previous section, input-saving technical change is negative input-using technical change.

⁶² β can be of either sign but if all the h_c have the same sign then β has this sign. A similar comment applies to γ .

$$\sum_{m=1}^M P_{1m} X_{1m} \text{ subject to (39.3) .} \quad (39.4)$$

In the second step, the firm chooses its inputs of labour, capital and land ($X_c, c=1, 2, 3$) to minimize

$$\sum_{c=1}^3 P_c X_c \text{ subject to (39.2) ,} \quad (39.5)$$

where P_1 , the overall price of labour defined by

$$P_1 = \sum_{m=1}^M P_{1m} \frac{X_{1m}}{X_1} \text{ ,} \quad (39.6)$$

with the X_{1m} s on the RHS of (39.6) being the solution to problem (39.4) and X_1 being part of the solution to (39.5). As required by problem (39.5), P_1 is independent of X_1 . This is because under (39.3), X_1 is a linearly homogeneous function of the X_{1m} s, implying that X_{1m}/X_1 determined in (39.4) depends only on the P_{1m} s, and not on X_1 . Under the definition in (39.6), P_1 is the minimum cost per unit of labour achievable at the given occupational wage rates (P_{1m}).

The first-order conditions for a solution of (39.4) can be expressed as

$$\lambda X_{1m}^{k_m-1} / (X_1 B_{1m})^{k_m} = P_{1m} \text{ ,} \quad m=1, \dots, M \quad (39.7)$$

plus (39.3).

In percentage-change form, these first-order conditions are:

$$\lambda + (k_m - 1)x_{1m} - k_m(x_1 + b_{1m}) = P_{1m} \text{ ,} \quad m=1, \dots, M \quad (39.8)$$

and

$$\sum_{m=1}^M S_{1m} x_{1m} = x_1 + \sum_{m=1}^M S_{1m} b_{1m} \text{ ,} \quad (39.9)$$

where λ , b_{1m} and the x 's and p 's are percentage changes in variables denoted by the corresponding upper case symbols, and

$$S_{1m} = \frac{(X_{1m}/X_1 B_{1m})^{k_m}}{\sum_q (X_{1q}/X_1 B_{1q})^{k_q}} \text{ ,} \quad m=1, \dots, M. \quad (39.10)$$

As can be established by multiplying (39.7) through by X_{1m} , S_{1m} is the share of occupation m in labour costs.

Following steps similar to those that took us from (37.6) and (37.7) to (37.16), we can manipulate (39.8) and (39.9) to obtain

$$x_{1m} = x_1 - \sigma_{1m} \left(P_{1m} - \sum_{q=1}^M S_{1q}^* P_{1q} \right) + b_{1m} - \sigma_{1m} \left(b_{1m} - \sum_{q=1}^M S_{1q}^* b_{1q} \right)$$

$$\text{for } m=1,\dots,M, \quad (39.11)$$

where σ_{1m} is the positive parameter defined by

$$\sigma_{1m} = 1/(1-k_m), \quad m=1,\dots,M \quad (39.12)$$

and S_{1q}^* is the modified cost-share defined by

$$S_{1q}^* = \sigma_{1q} S_{1q} / \sum_{m=1}^M \sigma_{1m} S_{1m}, \quad q=1,\dots,M. \quad (39.13)$$

Similarly, on solving the optimization problem (39.5), we obtain

$$x_s = z - \theta_s \left(p_s - \sum_{c=1}^3 S_c^* p_c \right) + b_s - \theta_s \left(b_s - \sum_{c=1}^3 S_c^* b_c \right) \quad \text{for } s=1,\dots,3, \quad (39.14)$$

where θ_s is the positive parameter defined by

$$\theta_s = 1/(1 - h_s) \quad s=1,\dots,3 \quad (39.15)$$

and S_c^* is the modified cost-share defined by

$$S_c^* = \theta_c S_c / \sum_{v=1}^3 \theta_v S_v, \quad c=1,\dots,3. \quad (39.16)$$

Finally, from (39.6) and (39.9) we derive an equation for the percentage change in the overall price of labour:

$$p_1 = \sum_{m=1}^M p_{1m} S_{1m} + \sum_{m=1}^M b_{1m} S_{1m}. \quad (39.17)$$

To move from (39.14), (39.11) and (39.17) to the MONASH equations E_{labind} , $E_{cap_at_t}$, E_{p1land} , $E_{x1laboi}$ and E_{p1lab} , we start by writing b_{1m} , $m=1,\dots,M$ and b_s , $s=1,\dots,3$ in terms of other variables (a 's and twist) as follows:

$$b_{1m} = a_{1m} - \sum_{q=1}^M S_{1q} a_{1q}, \quad m=1,\dots,M, \quad (39.18)$$

and

$$b_s = a_s + \psi_s \text{ twist} \quad s=1,\dots,3 \quad (39.19)$$

where ψ_s , $s=1,\dots,3$, satisfy the equations:

$$\psi_s - \theta_s \left(\psi_s - \sum_{c=1}^3 S_c^* \psi_c \right) = (C_{1s} V_2 - C_{2s} V_1), \quad s=1,\dots,3 \quad (39.20)$$

and

C_{1i} is a parameter with value 1 for $i=1$, otherwise 0;

C_{2i} is a parameter with value 1 for $i=2$, otherwise 0; and

V_1 and V_2 are coefficients representing the shares of labour and capital in the combined cost of labour and capital.

Next we substitute from (39.18) and (39.19) into (39.14), (39.11) and (39.17) and use (39.20) to eliminate ψ_s . Expressing a_1 , a_2 and a_3 as

$$a_1 = a_1 + a_1\text{prim} + a_1\text{primgen} + a_1\text{lab} + a_1\text{labgen} \quad , \quad (39.21)$$

$$a_2 = a_1 + a_1\text{prim} + a_1\text{primgen} + a_1\text{cap} \quad \text{and} \quad (39.22)$$

$$a_3 = a_1 + a_1\text{prim} + a_1\text{primgen} + a_1\text{land} \quad , \quad (39.23)$$

and noticing from (39.16) that the S_c^* sum to one so that a_1 , $a_1\text{prim}$ and $a_1\text{primgen}$ cancel out of the technology substitution term in (39.14) [the last term on the RHS], we obtain E_{labind} , $E_{\text{cap_at_t}}$, E_{pland} , E_{xllaboi} and E_{pllab} in simplified notation.

We conclude that these MONASH equations arise from cost minimization subject to CRESH constraints [problem (39.1) - (39.3)], with input-using technical changes (b_{1m} , b_s) restricted according to (39.18) - (39.20).

Restrictions (39.18) - (39.20) do not significantly limit the scope of MONASH. Restriction (39.18) means that a_{1q} can be used to impose only a cost-neutral occupation-q-using technical change, that is technical change which uses q and is offset by technical changes which save on all other occupations. However, if we wish to impose a pure occupation-q-using technical change, then we can combine an a_{1q} shock with an a_1 shock.⁶³

The only issue raised by restrictions (37.19) and (37.20) is the existence of values for ψ_s , $s=1,\dots,3$. If θ_s were one for all s ,⁶⁴ then we would not be able to find values for ψ_s , $s=1,\dots,3$ to satisfy (39.20). In this case, E_{labind} , $E_{\text{cap_at_t}}$ and E_{pland} would not be derivable from the cost minimization problem (39.1) - (39.3) with technical change recognized only through movements in the B_s . In general, to justify E_{labind} , $E_{\text{cap_at_t}}$ and E_{pland} via (39.1) - (39.3), we need to assume that the matrix C , given by

$$C = I - \hat{\theta}(I - S^*) \quad , \quad (39.24)$$

is non-singular, where $\hat{\theta}$ is the diagonal matrix whose i th diagonal element is θ_i , and S^* consists of three rows, each being S_1^*, S_2^*, S_3^* . The assumption that C is

⁶³ An a_{1q} shock of one per cent combined with an a_1 shock of S_{1q} per cent simulates occupation-q-using technical change of one per cent.

⁶⁴ We ruled out this case by assuming that h_c in (39.2) is not precisely zero. However, at the cost of some extra algebra, (39.2) could be generalized to embrace the Cobb-Douglas function. In this case θ_s would be one for all s .

non-singular is harmless. If C were singular, then a small change in $\hat{\theta}$ would produce a new model, arbitrarily close to the initial model, in which C is non-singular.

40. Appendix: the derivation of equations for household demands, discussed in section 22

We assume that the household sector chooses $X3(i)$ and $X3CS(i,s)$, $i \in \text{COM}$, $s \in \text{SOURCE}$, to maximize

$$\sum_i \Delta_i \ln(X3(i)/Q - \Gamma_i) \quad (40.1)$$

subject to

$$X3(i) = \text{CES}[X3CS(i, \text{dom})/B(i, \text{dom}); X3CS(i, \text{imp})/B(i, \text{imp})], \quad i \in \text{COM}, \quad (40.2)$$

and

$$\sum_i \sum_s P3CS(i,s)X3CS(i,s) = C, \quad (40.3)$$

where

$X3CS(i,s)$ is consumption of (i,s) ;

$X3(i)$ is overall consumption of i ;

Q is the number of households;

CES denotes a CES function;

$P3CS(i,s)$ is the price to households of good i from source s ; and

Δ_i, Γ_i and $B(i,s)$ are preference or taste change variables with $B(i,s)$ and Δ_i being positive, and with the Δ_i the summing to one over all i .

In subsections 40.1 to 40.3 we show that (40.1) – (40.3), together with the restrictions placed by MONASH on movements in preference variables, lead to E_{x3} , E_{x3cs} and E_{p3} in subsection 18.8e with the price and expenditure elasticities set by the relevant formulas in subsection 18.5k.

40.1. Derivation of E_{x3} and interpretation of the taste-change variables $a3com$

We solve the utility maximization problem (40.1)-(40.3) in two stages. First, we choose $X3(i)$, $i \in \text{COM}$ to maximize (40.1) subject to

$$\sum_i P3(i)X3(i) = C, \quad (40.4)$$

where $P3(i)$ is the minimum cost to households of obtaining a unit of good i .⁶⁵ This first stage optimization produces the demand system:

$$X3(i)/Q = \Gamma_i + [\Delta_i / P3(i)] \{ (C/Q) - \sum_j \Gamma_j P3(j) \}, \quad i \in \text{COM} . \quad (40.5)$$

In percentage-change and change⁶⁶ form, (40.5) can be written as:

$$\begin{aligned} x3(i) - q &= \gamma_i - (1/F) [\Delta_i / S_i] (\delta_i - p3(i)) \\ &+ [\Delta_i / S_i (i)] (c - q) - [\Delta_i / S_i] \sum_j S_j \gamma_j - (1/F) [\Delta_i / S_i] \sum_j (\Delta_j + FS_j) p3(j), \end{aligned} \quad (40.6)$$

where

$x3(i)$, q , δ_i , c and $p3(i)$ are percentage changes in $X3(i)$, Q , Δ_i , C and $P3(i)$;

S_i is the share of good i in the household budget;

F is the Frisch⁶⁷ coefficient defined by

$$F = -(C/Q) / \{ C/Q - \sum_j \Gamma_j P3(j) \} , \quad (40.7)$$

and γ_j is 100 times the change in Γ_j divided by $X3(j)/Q$, that is,

$$\gamma_j = 100 \text{del} \Gamma_j / (X3(j)/Q) , \quad j \in \text{COM} . \quad (40.8)$$

In deriving (40.6), we eliminated Γ_i by using

$$\Gamma_i = X3(i)/Q + (\Delta_i / P3(i)) (C/Q) / F , \quad (40.9)$$

which implies $P3(i) \Gamma_i = (C/Q) (S_i + \Delta_i / F)$. (40.10)

From (40.6), we obtain

$$x3(i) - q = \varepsilon_i (c - q) + \sum_j \eta_{ij} p3(j) + (\gamma_i - \varepsilon_i \sum_j S_j \gamma_j) - (1/F) \varepsilon_i \delta_i , \quad i \in \text{COM} , \quad (40.11)$$

where ε_i , the expenditure elasticity of demand for good i , is given by

$$\varepsilon_i = \Delta_i / S_i , \quad i \in \text{COM}, \quad (40.12)$$

⁶⁵ The $P3$ s are independent of the $X3$ s because the CES function (40.2) is homogeneous of degree 1 in $X3CS(i,s)$, see SOURCE.

⁶⁶ We avoid percentage changes in Γ_i because it is a variable that can pass through zero.

⁶⁷ Named for Frisch (1959) who together with Houthakker (1960) found that under an additive utility function, cross-price elasticities (η_{ij}), expenditure elasticities (ε_i) and budget shares (S_j) are connected by the equation

$$\eta_{ij} = -\varepsilon_i S_j [1 + \varepsilon_j / F] \quad i \neq j ,$$

where F is independent of i and j . Under the additive utility function (40.1), F is given by (40.7).

and η_{ij} , the elasticity of demand for good i with respect to the price of j , is given by

$$\eta_{ij} = \text{KD}(i,j)(\varepsilon_i/F) - \varepsilon_i[S_j + \Delta_j/F] \quad , \quad i,j \in \text{COM} . \quad (40.13)$$

In (40.13), $\text{KD}(i,j)$ is the Kronecher delta: it is one if $i=j$ and zero otherwise.

On looking at the TABLO formulas in subsection 18.5k for $\text{EPS}(i)$ and $\text{ETA}(i,j)$, we see that the expenditure and price elasticities in E_x3 are set according to (40.12) and (40.13).

Now to derive E_x3 , all that is required is to check that $E_deltapc$ and E_d_gamma reduce the taste-change term in (40.11) to $a3com(i) - ave_a3com$. Using simplified notation, we rewrite E_ave_a3com , $E_deltapc$ and E_d_gamma as

$$\bar{a} = \sum_j S_j a_j \quad (40.14)$$

$$\delta_i = a_i - \sum_k \Delta_k a_k \quad (40.15)$$

and

$$\gamma_i = (1 + \varepsilon_i/F) (a_i - \bar{a}) . \quad (40.16)$$

Our task is to show that

$$\gamma_i - \varepsilon_i \sum_j S_j \gamma_j - (1/F) \varepsilon_i \delta_i = a_i - \bar{a} , \quad i \in \text{COM} . \quad (40.17)$$

To establish (40.17), we substitute (40.15) and (40.16) into the LHS obtaining

$$\text{LHS} = (1 + \varepsilon_i/F) (a_i - \bar{a}) - \varepsilon_i \sum_j S_j (1 + \varepsilon_j/F) (a_j - \bar{a}) - (1/F) \varepsilon_i (a_i - \sum_k \Delta_k a_k) . \quad (40.18)$$

By using (40.12) and (40.14) and recalling that the sum of the Δ_i is one, we find that LHS simplifies, as required, to $a_i - \bar{a}$.

Thus we have shown that E_x3 , together with the related elasticity formulas in subsection 18.5k, is derived from (40.1) – (40.3) with the movements (δ_i and γ_i) in the 2G preference variables Δ_i and Γ_i restricted by the G variables a_i according to (40.14) to (40.16). What are the implication of these restrictions for the movements in Δ_i and Γ_i ?

For answering this question it is convenient to assume that Γ_i is not zero. Then (40.8) implies that

$$\% \text{ change } \Gamma_i = (\gamma_i/\Gamma_i)(X3(i)/Q) , \quad (40.19)$$

and via (40.9), (40.12) and (40.16) we find that

$$\% \text{ change } \Gamma_i = a_i - \bar{a} . \quad (40.20)$$

Normally, we expect \bar{a} to be close to $\sum_k \Delta_k a_k$, implying [via (40.15) and (40.20)] that the percentage changes in Γ_i and Δ_i are approximately the same.

Thus, when we impose a 10 per cent shift in consumer preferences in favour of good i ($a_i - \bar{a} = 10$), then providing Γ_i is not zero, we are assuming that there is an increase of approximately 10 per cent in Δ_i combined with an increase of 10 per cent in Γ_i .

The final issue to be considered in this subsection is the updating of Γ_i and Δ_i . This is straight forward for Δ_i [DELTA(i) in the TABLO code]. Values are read from the data file EXTRA and updated using `deltapc`.⁶⁸

For Γ_i we have followed a different approach. Because Γ_i involves quantity units, it is awkward to transfer estimates of its level from econometric studies into our model. Instead we use an estimate of F [FRISCH in the TABLO code]. This is read from the data file EXTRA. Then, as we have already seen, expenditure and price elasticities are computed in subsection 18.5k using formulas not involving Γ_i . The closest the TABLO code comes to an explicit representation of Γ_i is in the coefficient SS3COM(i), computed in subsection 18.5k. In terms of the notation of the present section,

$$SS3COM(i) = S_i + \Delta_i / F . \quad (40.21)$$

By summing (40.21) over all i , we obtain

$$F = -1/(1 - \sum_j SS3COM(j)) . \quad (40.22)$$

We also see [via (40.10)] that (40.21) implies:

$$SS3COM(i) = P3(i) \Gamma_i / (C/Q) . \quad (40.23)$$

From (40.22), (40.23) and (40.8), we find that when $P3(i)$, Γ_i , C and Q change, the new value of F is given by

$$F_{new} = -100/[100 - \sum_j \{SS3COM(j)_{old}(100 + p3(j) - c + q) + S_{j,old} \gamma_j\}] . \quad (40.24)$$

In the TABLO code (subsection 18.7), we use (40.24) in updating FRISCH.

40.2. Derivation of E_x3cs and interpretation of the taste-change variables $a3cs$ and $twist_src$

In the second stage of the consumer utility maximization we solve, for each $i \in COM$, a problem of the form:

choose $X3CS(i,s)$, $s \in SOURCE$,

to minimize $\sum_s P3CS(i,s)X3CS(i,s)$ subject to (40.2).

⁶⁸ The Δ_i s should sum to one. As can be seen in subsection 18.5k of the TABLO code, we make sure that this happens by normalizing the values read for DELTA(i) by Formula (Initial) statements.

That is, we determine the combination of domestic and imported good i which generates the household sector's overall consumption of i [$X_3(i)$] at minimum cost.

This second stage problem has the same form as the cost-minimization problem (38.1)-(38.2). By following the steps in section 38, we arrive at an equation similar to (38.11). We write this equation as

$$\begin{aligned} x_{3cs}(i,s) = & x_3(i) - \text{SIGMA}_3(i) [p_{3cs}(i,s) - \sum_t S(i,t) p_{3cs}(i,t)] \\ & + b(i,s) - \text{SIGMA}_3(i) [b(i,s) - \sum_t S(i,t) b(i,t)], \quad s \in \text{SOURCE} \end{aligned} \quad (40.25)$$

where

$x_{3cs}(i,s)$, $x_3(i)$, $p_{3cs}(i,s)$ and $b(i,s)$ are the percentage changes in the corresponding uppercase variables;

$\text{SIGMA}_3(i)$ is the household sector's elasticity of substitution between the two sources of good i ; and

$S(i,t)$ is the share of good (i,t) in household expenditure on i ($\text{SOURCE_SH}_3(i,t)$ in the TABLO code).

Provided $\text{SIGMA}_3(i)$ is not precisely one, we can arrive at $E_{x_{3cs}}$ by restricting the (i,s) -using changes in household preferences according to

$$b(i,s) = a_{3cs}(i,s) - \sum_t S(i,t) a_{3cs}(i,t) + C(i,s) * \text{twist_src}(i), \quad s \in \text{SOURCE} \quad (40.26)$$

and

$$C(i,s) = -(\text{SOURCEDOM}(s) - S(i,\text{dom})) / (1 - \text{SIGMA}_3(i)), \quad s \in \text{SOURCE}, \quad (40.27)$$

and then substituting into (40.25).⁶⁹ Restrictions (40.26) and (40.27) not only underlie E_{x_3} but also have the desirable implication that

$$\sum_t S(i,t) b(i,t) = 0, \quad (40.28)$$

which is necessary and sufficient for preserving the intuitively natural relationship

$$\sum_t S(i,t) x_{3cs}(i,t) = x_3(i). \quad (40.29)$$

Continuing to assume that $\text{SIGMA}_3(i)$ is not equal to one, we see that the "a" variables in $E_{x_{3cs}}$ can now be interpreted as follows: $a_{3cs}(i, \text{dom}) = 1$ implies a good- (i, dom) -using change in preferences of $S(i, \text{imp})$ per cent combined with a good- (i, imp) -using change in preferences of $-S(i, \text{dom})$ per cent. The same change in preferences is generated with $\text{twist_src}(i) = -(1 - \text{SIGMA}_3(i))$. Similarly, $a_{3cs}(i, \text{imp}) = 1$ implies a good- (i, imp) -using change in preferences of $S(i, \text{dom})$ per cent combined with a good- (i, dom) -using change in preferences of $-S(i, \text{imp})$ per cent. This change in preferences can be generated by $\text{twist_src}(i) =$

⁶⁹ As defined in subsection 20.3, $\text{SOURCEDOM}(s)$ is 1 for $s = \text{dom}$ and 0 for $s = \text{imp}$.

(1-SIGMA3(i)). Thus, the "a" variables and twist_src give alternative methods of imposing identical twists in household preferences between domestic and imported goods.

If SIGMA3(i) is equal to one, then non-zero values for a3cs(i,s) have no effect. To impose shifts in domestic/import preferences we must use twist_src(i). However, when we do this, there is no easy interpretation in terms of the underlying domestic/import preference function, (40.2), of the taste changes imposed.

40.3. Derivation of E_p3

Finally, we note that the minimum cost to households of a unit of good i [P3(i)] is given by

$$P3(i) = \sum_s P3CS(i,s)X3CS(i,s) / X3(i) \quad , \quad (40.30)$$

where X3CS(i,s) and X3(i) are the utility maximizing consumption levels for good (i,s) and good i. From (40.30) we obtain

$$p3(i) = \sum_s (p3cs(i,s) + x3cs(i,s))S(i,s) - x3(i) \quad . \quad (40.31)$$

By using (40.29) we can simplify (40.31) to E_p3.

41. Appendix: the theory underlying the export demand equations (E_x4_TRADEXP), discussed in section 23

This appendix provides theoretical background for the export demand equations, E_x4_TRADEXP, in subsection 18.8f. The issues of particular concern are: (a) the interpretation of the parameter EXP_ELAST(i) and the use in E_x4_TRADEXP of f.o.b. prices [pe(i)] for exports rather than purchasers' prices in foreign countries; and (b) the interpretation of horizontal [feq(i) + feq_general] and vertical [fep(i)] shifts in export demand curves.

Our theory underlying E_x4_TRADEXP starts with the following specification of behaviour by foreign users of Australian good i: foreigners choose $X_a(i)$ and $X_o(i)$ to minimize

$$P_a(i)X_a(i) + P_o(i)X_o(i) \quad (41.1)$$

$$\text{subject to} \quad R(i) = \text{CES}(X_a(i)/B_a(i), X_o(i)/B_o(i)) \quad (41.2)$$

where

$P_a(i)$ and $P_o(i)$ are the purchasers' prices in foreign countries of good i from Australia and good i from alternative sources;

$X_a(i)$ and $X_o(i)$ are foreign demands for good i from Australia and good i from alternative sources;

$B_a(i)$ and $B_o(i)$ are variables allowing for changes in foreign preferences for good i from Australia and good i from alternative sources;

$R(i)$ is foreign requirements for good i ; and

$CES(\dots)$ denotes CES function.

From problem (41.1)-(41.2)⁷⁰, we obtain

$$x_a(i) = r(i) - \phi(i)S_o(i)(p_a(i) - p_o(i)) + b_a(i) - \phi(i)S_o(i)(b_a(i) - b_o(i)) \quad (41.3)$$

where

$\phi(i)$ is the foreign elasticity of substitution between Australian and alternative sources of supply of good i ;

$S_o(i)$ is the share of non-Australian sources in foreign purchases of good i ; and

$x_a(i)$, $r(i)$, $p_a(i)$, $p_o(i)$, $b_a(i)$ and $b_o(i)$ are percentage changes in the variables denoted by the corresponding upper-case symbols.

Next we break $p_a(i)$ into two parts, associated with: (a) the percentage change [$pe(i)$] in the foreign-currency f.o.b. price; and (b) the percentage change [$q(i)$] in the unit charges (including transport costs and tariffs) separating Australian ports and foreign sites at which Australian good i is used. This gives

$$p_a(i) = S_{fob}(i) pe(i) + [1 - S_{fob}(i)]q(i) \quad (41.4)$$

where $S_{fob}(i)$ is the share of the f.o.b. price of Australian good i in its purchaser's price in foreign countries.

The final component in our theory is an equation for percentage movements [$r(i)$] in world requirements for good i . We assume that world demand for i responds to movements in the world price for i relative to a general index of the prices of all goods. The world price of i is an average of the prices of Australian good i and good i from other sources. Thus, we write:

$$r(i) = g(i) + \eta(i)[(1 - S_o(i))p_a(i) + S_o(i)p_o(i) - winf] \quad (41.5)$$

where

$g(i)$ is the horizontal shift in the world demand curve for good i ;

$\eta(i)$ is the elasticity of world demand for good i ; and

$winf$ is the world rate of inflation.

On substituting from (41.4) and (41.5) into (41.3), and bringing the notation into line with that in the TABLO code, we obtain $E_x4_TRADEXP$:

$$x4(i) = EXP_ELAST(i)[pe(i) - fep(i)] + feq(i) + feq_general \quad (41.6)$$

⁷⁰ We have not set out the algebra which takes us from (41.1)-(41.2) to (41.3). It is the same as that which took us from (38.1)-(38.2) to (38.11). However, (41.3) is set out differently from (38.11) in two ways. First, in (41.3) we are concerned with the demand for the product from only one of the sources (Australia). Second, in the price and technology substitution terms in (41.3), we have recognized that $1 - S_a(i) = S_o(i)$ where $S_a(i)$ is the Australian share in foreign purchases of good i .

where

$$x_4(i) = x_a(i) \quad (41.7)$$

$$\text{EXP_ELAST}(i) = \{\eta(i)[1 - S_o(i)] - \phi(i)S_o(i)\}S_{fob}(i) \quad , \quad (41.8)$$

$$\text{feq}(i) + \text{feq_general} = g(i) + b_a(i) - \phi(i)S_o(i)[b_a(i) - b_o(i)] \quad , \quad (41.9)$$

and

$$\begin{aligned} \text{fep}(i) = & -[(1 - S_{fob}(i))/S_{fob}(i)]q(i) \\ & - [\{\eta(i)S_o(i) + \phi(i)S_o(i)\}/\text{EXP_ELAST}(i)]p_o(i) + [\eta(i)/\text{EXP_ELAST}(i)]\text{winf} \quad . \end{aligned} \quad (41.10)$$

MONASH does not contain equations or data to determine the coefficients on the RHS of (41.8). Consequently, as mentioned in section 23, we treat $\text{EXP_ELAST}(i)$, the foreign elasticity of demand for good i , as a parameter. In most foreign markets, Australian commodities account for only a small share of sales [$S_o(i)$ is close to 1] and face considerable competition from substitutes produced in other countries [$\phi(i)$ is large]. If $S_{fob}(i)$ is 0.7, $S_o(i)$ is close to 1 and $\phi(i) = 6$, then we obtain our typical value for $\text{ELAST_EXP}(i)$ of -4.

For a few commodities (most notably wool), the Australian variety is both distinctive [$\phi(i)$ is low] and occupies a major share of foreign markets [$S_o(i)$ is comparatively low]. For such commodities, values for $\text{ELAST_EXP}(i)$ smaller in absolute value than 4 are appropriate. For example, if $\phi(i) = 3.2$, $S_o(i) = 0.5$, $S_{fob}(i) = 0.7$ and $\eta(i) = -0.5$, then $\text{ELAST_EXP}(i) = -1.3$. This is the value we are currently using in MONASH for the export demand elasticity for wool.

In (41.9) and (41.10) we have treated changes in quantity and preference variables as horizontal shifts in export demand curves and changes in price variables as vertical shifts. Other allocations of variables between these two categories could have been chosen. Given our choice, horizontal shifts ($\text{feq}(i) + \text{feq_general}$) in the export demand curve for Australia's good i are associated with changes in world demand for i [$g(i)$] and shifts in foreign preferences towards or away from Australia's variety of good i [$b_a(i), b_o(i)$]. Vertical shifts [$\text{fep}(i)$] are associated with: (a) movements in unit costs [$q(i)$] separating Australian ports of exit and foreign sites of use; (b) movements in foreign markets in the prices [$p_o(i)$] of products competing with Australia's good i ; and (c) movements in the world's overall price level [winf]. With typical values for $S_{fob}(i), \phi(i), \eta(i)$ and $S_o(i)$, say 0.7, 6, -0.5 and 0.95, the values of the coefficients on $q(i), p_o(i)$ and winf in (41.10) are -0.429, 1.304 and 0.125. These coefficient values imply that: a one per cent increase in transport, tariff and other separating costs per unit of export of good i from Australia generates a vertical downward

shift in the export demand curve for Australia's good i of 0.429 per cent; a one per cent increase in world prices of products competitive with Australia's good i produces a vertical upward shift in the export demand curve for Australia's good i of 1.304 per cent; and a one per cent overall increase in the world price level produces a vertical upward shift in Australia's export demand curve for good i of 0.125 per cent.

In many MONASH applications it is appropriate to set the vertical shift $fep(i)$ in the export demand curve for good i at the world rate of inflation. This is justified if the percentage movements in $q(i)$, $p_o(i)$ and $winf$ are the same. [Notice in (41.10) that the sum of the coefficients on $q(i)$, $p_o(i)$ and $winf$ is one.] If we wish to simulate the effects of movements in either $q(i)$ or $p_o(i)$ which are different from the world rate of inflation, then we can use (41.10), outside MONASH, to compute the appropriate value for $fep(i)$ to be applied as a shock inside MONASH. Alternatively, we could include (41.10) in the model with $q(i)$, $p_o(i)$ and $winf$ as explicit exogenous variables, and $fep(i)$ as an endogenous variable.

42. Appendix: zero pure profits in production; the definition of technical change by industry; the GDP identity; and the income-side definition of the percentage change in real GDP

In this section we: (a) derive the zero-pure-profits and the technical change equations, E_z , E_a_{JP} and E_a_{UP} , in subsection 18.8j; and (b) establish that MONASH satisfies the GDP identity in both nominal and real terms.

As discussed in section 27, the MONASH equations for zero pure profits in production (E_z) take the form

$$p_0(j) = a(j) + p_1(j) + \frac{\Pi(j)}{(2 - \Pi(j))} * \pi(j) \quad j \in IND, \quad (42.1)$$

where

$p_0(j)$ is a revenue-weighted average of the percentage changes in the prices received by industry j for its products;

$a(j)$ is an aggregation [defined in E_a_{JP} and E_a_{UP}] of technical changes affecting industry j 's production function;

$p_1(j)$ is a weighted average of the percentage changes in the prices paid by industry j for its inputs with the weights being input shares in total input costs (i.e., costs excluding production taxes); and

$\Pi(j)$ and $\pi(j)$ are the level and percentage change of the power of the production tax.

In subsections 42.1 (for joint-product industries) and 42.2 (for unique-product industries), we show how the MONASH equations of the form (42.1), together with the MONASH equations for $a(j)$, are equivalent to the more basic zero-pure-profits condition:

$$\text{revenue}(j) = \text{costs}(j), \quad j \in \text{IND} \quad (42.2)$$

where $\text{revenue}(j)$ and $\text{costs}(j)$ are percentage changes in j 's revenue and costs.

The demonstration is long and tedious because of the occurrence in MONASH of multi-industry commodities and multi-commodity industries, and the detailed treatments of taxes, margins and purchasers' prices. In working through the algebra it is helpful to keep in mind that our strategy is as follows. First we show that (42.2) can be represented as:

$$p_0(j) + \sum_i H(i,j) * x_0(i,j) = p_1(j) + \sum_q S(q,j) * x_1(q,j) + \frac{\Pi(j)}{(2 - \Pi(j))} * \pi(j), \quad (42.3)$$

where

$H(i,j)$ is the share of j 's revenue generated by sales of commodity i ;

$S(q,j)$ is the share of j 's input costs (costs excluding production taxes) accounted for by input q ;

$x_0(i,j)$ is the percentage change in j 's output of i ; and

$x_1(q,j)$ is the percentage change in j 's inputs of q .

Then, on comparing (42.3) with (42.1), we see that one way to demonstrate their equivalence is to show that

$$a(j) = - \sum_i H(i,j) * x_0(i,j) + \sum_q S(q,j) * x_1(q,j) \quad (42.4)$$

We eventually derive (42.4) by going to the commodity-supply and input-demand equations in MONASH and showing that these imply:

$$- \sum_i H(i,j) * x_0(i,j) = - z(j) + a_0(j) \quad (42.5)$$

$$\sum_q S(q,j) * x_1(q,j) = z(j) + a_1(j) \quad (42.6)$$

and

$$a(j) = a_0(j) + a_1(j) \quad (42.7)$$

where

$z(j)$ is the percentage change in the activity level in industry j ; and

$a_0(j)$ and $a_1(j)$ are amalgams of technical change terms.

Because we always start MONASH simulations with balanced input-output data, an implication of the work in subsections 42.1 and 42.2 is that the levels form of (42.2) holds throughout MONASH multi-step simulations, i.e.

$$\text{REVENUE}(j) = \text{COSTS}(j) \quad j \in \text{IND}. \quad (42.8)$$

By combining (42.8) with market-clearing conditions, we show in subsection 42.3 that MONASH will always satisfy the nominal GDP identity both in levels and percentage changes, i.e., we show that throughout MONASH computations,

$$\text{GDPEX} = \text{GDPIN} \quad \text{and} \quad \text{gdpexp} = \text{gdpinc} \quad (42.9)$$

where

GDPEX and GDPIN are GDP calculated in MONASH from the expenditure and income sides of the national accounts, and

gdpexp and gdpinc are percentage changes in GDPEX and GDPIN.

Finally, in subsection 42.4, we extend the algebra underlying (42.5) to (42.7) to validate the income-side definition of real GDP given in $E_gdprealinc$ (subsection 18.8l). We prove that $gdprealinc$ defined in this equation is equal to the percentage change in real GDP defined in $E_gdpreal$, that is we show that MONASH satisfies the GDP identity in real terms.

42.1. Derivation of E_z for joint-production industries

For joint-production industries ($j \in IND_JP$), the zero pure-profits condition, (42.8), can be written as:

$$\sum_{i \in COM_JP} MAKE(i,j) = \sum_{i \in COM} \sum_s PURCHVAL1(i,s,j) + LABIND_J(j) + CAPITAL(j) + LAND(j) + OTHCOST(j) + TAX0(j), \quad j \in IND_JP, \quad (42.10)$$

where the coefficients MAKE, PURCHVAL1, etc. are defined in cross-hatched comments in subsection 18.3. The LHS of (42.10) is industry j 's revenue calculated as the sum of j 's sales over all commodities i in the set COM_JP . The RHS of (42.10) is j 's costs calculated as the sum of the costs to j of: intermediate inputs; labour; capital rentals; land rentals; other costs; and production taxes.

Expressing each sale and input cost in (42.10) as the product of a price and quantity, and recognizing that production taxes are a rate (a power minus one) times revenue, we obtain after some rearrangement another representation of (42.8) for joint-product industries:

$$(2-POW_TAX0(j)) * \sum_{i \in COM_JP} P0DOM(i) * X0CI(i,j) = \sum_{i \in COM} \sum_s P1CSI(i,s,j) * X1CSI(i,s,j) + \sum_{qq \in OCC} P1LABOI(qq,j) * X1LABOI(qq,j) + P1CAP(j) * CAP_AT_T(j) + P1LAND(j) * N(j) + P1OCT(j) * X1OCT(j), \quad j \in IND_JP, \quad (42.11)$$

where the meaning of the notation can be deduced from cross-hatched comments in subsection 18.6. For example, in subsection 18.6 we see that $p0dom(i)$ is the percentage change in the basic price of commodity i . Thus, in (42.11), $P0DOM(i)$ is the basic price of commodity i .

In percentage-change form, (42.11) can be written as:

$$\sum_{i \in COM_JP} H0CI(i,j) * p0dom(i) = \psi(j) + [1/(COSTS(j) - TAX0(j))] * \{ \sum_i \sum_s PURCHVAL1(i,s,j) * p1csi(i,s,j)$$

$$\begin{aligned}
 &+ \sum_{qq} \text{LABOCCIND}(qq,j) * p1laboi(qq,j) + \text{CAPITAL}(j) * p1cap(j) \\
 &+ \text{LAND}(j) * p1land(j) \qquad \qquad \qquad \dots \text{equation continues} \\
 &+ 100 * \text{ROTHCOST}(j) * del_p1oct(j) + \frac{\text{POW_TAX0}(j)}{(2 - \text{POW_TAX0}(j))} * powtax0(j) , \\
 & \qquad \qquad \qquad j \in \text{IND_JP}, \qquad (42.12)
 \end{aligned}$$

where

$$\begin{aligned}
 \psi(j) = & - \sum_{i \in \text{COM_JP}} \text{HOCI}(i,j) * x0ci(i,j) \\
 & + [1 / (\text{COSTS}(j) - \text{TAX0}(j))] * \{ \sum_i \sum_s \text{PURCHVAL1}(i,s,j) * x1csi(i,s,j) \\
 & + \sum_{qq} \text{LABOCCIND}(qq,j) * x1laboi(qq,j) + \text{CAPITAL}(j) * cap_at_t(j) \\
 & + \text{LAND}(j) * n(j) + \text{OTHCOST}(j) * x1oct(j) \} \qquad \text{for } j \in \text{IND_JP}. \qquad (42.13)
 \end{aligned}$$

All the coefficients in (42.12) and (42.13) are defined in subsection 18.3. In terms of the strategy outlined earlier for establishing E_z , (42.12) and (42.13) taken together are the equivalent of (42.3).

Our aim in this subsection is to show that (42.12) and (42.13) are implied by E_z , $j \in \text{IND_JP}$, together with all the equations in subsection 18.8 excluding E_z for $j \in \text{IND}$, and that E_z , $j \in \text{IND_JP}$, is implied by (42.12) and (42.13) together with all the equations in subsection 18.8 excluding E_z for $j \in \text{IND}$ i.e., we aim to show that

$$M + E_z(j \in \text{IND_JP}) \Rightarrow (42.12) \text{ and } (42.13) \qquad \text{(I)}$$

and

$$M + (42.12) \text{ and } (42.13) \Rightarrow E_z(j \in \text{IND_JP}), \qquad \text{(II)}$$

where M is the set of MONASH equations in subsection 18.8 excluding E_z for $j \in \text{IND}$. By establishing (I) and (II), we will have shown that

$$M + E_z(j \in \text{IND_JP}) \Leftrightarrow M + (42.12) \text{ and } (42.13) .$$

In other words, we will have shown in the context of the MONASH model that representing the zero-pure-profits conditions for joint-product industries as E_z ($j \in \text{IND_JP}$) is equivalent to representing them as (42.12) and (42.13).

On inspecting (42.12), and E_{p0ind} ⁷¹ and E_z in subsection 18.8j, we see that both (I) and (II) can be proved by showing that:

⁷¹ Recall that joint-product industries produce only joint-product commodities. Hence, for $j \in \text{IND_JP}$, $\sum_{i \in \text{COM_JP}} \text{HOCI}(i,j) * p0dom(i) = \sum_{i \in \text{COM}} \text{HOCI}(i,j) * p0dom(i)$.

$$M \Rightarrow \psi(j) = a(j) \text{ for all } j \in \text{IND_JP} \quad (42.14)$$

where $\psi(j)$ is defined by (42.13) and $a(j)$ is as defined in E_a_JP . In terms of the strategy outlined earlier for deriving E_z , establishing (42.14) corresponds to establishing (42.4). Our task for the remainder of this subsection is to establish (42.14).

This requires us to study each of the terms on the RHS of (42.13) starting with

$$\text{Term}(1) = - \sum_{i \in \text{COM_JP}} \text{HOCI}(i,j) * x0ci(i,j), \quad j \in \text{IND_JP}.$$

From E_x0ci [in subsection 18.8a], we find that

$$\begin{aligned} \text{Term}(1) = & - \sum_{i \in \text{COM_JP}} \sum_{cc \in \text{COMPCOM}} \text{HOCI}(i,j) * \text{CCPROD}(i,j,cc) * x0ccom(cc,j) \\ & + \sum_{i \in \text{COM_JP}} \text{HOCI}(i,j) * [a0com(i) + a0ci(i,j)] \quad , \quad j \in \text{IND_JP}. \end{aligned} \quad (42.15)$$

In deriving (42.15), we recognized that

$$\begin{aligned} \sum_i \sum_{cc} \text{HOCI}(i,j) * \text{CCPROD}(i,j,cc) * \text{SIGMA}(cc,j) * \\ [\beta(i) - \sum_{ii} \text{CCPROD}(i,j,cc) * \text{CCPROD}(ii,j,cc) * \text{SOC}(ii,j) * \beta(ii)] = 0, \quad j \in \text{IND_JP}, \end{aligned} \quad (42.16)$$

where $\beta(i)$ can be interpreted as either $p0dom(i)$ or $[-a0com(i) - a0ci(i,j)]$. In demonstrating (42.16), it is helpful to note from the formulas in subsection 18.5c that

$$\text{HOCI}(i,j) * \text{CCPROD}(i,j,cc) = \text{HOCC}(cc,j) * \text{SOC}(i,j) * \text{CCPROD}(i,j,cc) \quad (42.17)$$

and

$$\sum_i \text{CCPROD}(i,j,cc) * \text{SOC}(i,j) = \sum_i \text{CCPROD}(i,j,cc)^2 * \text{SOC}(i,j) = 1 \quad . \quad (42.18)$$

Next we note that $\sum_{cc} \text{HOCC}(cc,j) = 1$. Using this together with (42.17) and (42.18) and substituting from E_x0ccom [subsection 18.8a] into (42.15) gives

$$\begin{aligned} \text{Term}(1) = & -z(j) + a0ind(j) + \sum_{cc} \text{HOCC}(cc,j) * a0ccom(cc,j) \\ & + \sum_{i \in \text{COM_JP}} \text{HOCI}(i,j) * [a0com(i) + a0ci(i,j)] \quad . \end{aligned} \quad (42.19)$$

In deriving (42.19), we recognized that⁷²

$$\sum_{cc} \text{HOCC}(cc,j) * \text{SIG0CC}(cc,j) * [\beta(cc,j) - \sum_t \text{MH0CC}(t,j) * \beta(t,j)]$$

⁷² It is helpful to note from subsection 18.5c that $\sum_{cc} \text{HOCC}(cc,j) * \text{SIG0CC}(cc,j) = \text{SUMSIG}(j)$, and that $\text{SUMSIG}(j) * \text{MH0CC}(t,j) = \text{HOCC}(t,j) * \text{SIG0CC}(t,j)$.

$$\begin{aligned}
&= \sum_{cc} H0CC(cc,j)*SIG0CC(cc,j)*\beta(cc,j) - \text{SUMSIG}(j)* \sum_t MH0CC(t,j)*\beta(t,j) \\
&= \sum_{cc} H0CC(cc,j)*SIG0CC(cc,j)*\beta(cc,j) - \sum_t H0CC(t,j)*SIG0CC(t,j)*\beta(t,j) = 0 \quad ,
\end{aligned}$$

where $\beta(cc,j)$ can be interpreted as either $p0ccom(cc,j)$ or $a0ccom(cc,j)$. In terms of the strategy outlined earlier for establishing E_z , (42.19) corresponds to (42.5).

The second term on the RHS of (42.13) is

$$\text{Term}(2) = [1/(\text{COSTS}(j) - \text{TAX0}(j))] * \sum_i \sum_s \text{PURCHVAL1}(i,s,j) * x1csi(i,s,j). \quad (42.20)$$

Using E_x1csi [from subsection 18.8b], we obtain

$$\begin{aligned}
\text{Term}(2) &= [1/(\text{COSTS}(j) - \text{TAX0}(j))] * \{ \sum_i \text{TPURCHVAL1}(i,j) * [z(j) + a1(j) \\
&\quad + a1ci(i,j)] + \sum_i \sum_s \text{PURCHVAL1}(i,s,j) * a1csi(i,s,j) \\
&\quad + \sum_{i \in \text{TRANSERV}} \text{PURCHVAL1}(i, \text{imp},j) * f1_trans(j) \\
&\quad + \sum_{i \in \text{COMMUNICAT}} \text{PURCHVAL1}(i, \text{imp},j) * f1_commun(j) \} \quad . \quad (42.21)
\end{aligned}$$

In deriving (42.21), we recognized that

$$\begin{aligned}
&\sum_i \sum_s \text{PURCHVAL1}(i,s,j) * \text{SIGMA1}(i) \\
&\quad * \{ \beta(i,s,j) - \sum_t \text{SOURCE_SHR1}(i,t,j) * \beta(i,t,j) \} \\
&= \sum_i \text{TPURCHVAL1}(i,j) * \text{SIGMA1}(i) \\
&\quad * \{ \sum_s \text{SOURCE_SHR1}(i,s,j) * \beta(i,s,j) - \sum_t \text{SOURCE_SHR1}(i,t,j) * \beta(i,t,j) \} = 0,
\end{aligned}$$

where $\beta(i,s,j)$ can be interpreted as either $p1csi(i,s,j)$ or $a1csi(i,s,j)$, and that

$$\begin{aligned}
&\sum_i \sum_s \text{PURCHVAL1}(i,s,j) \\
&\quad * [\text{SOURCEDOM}(s) - \text{SOURCE_SHR1}(i, \text{dom},j)] * \text{twist_src}(i) \\
&= \sum_i \text{TPURCHVAL1}(i,j) * \text{twist_src}(i) * \sum_s \text{SOURCE_SHR1}(i,s,j) \\
&\quad * [\text{SOURCEDOM}(s) - \text{SOURCE_SHR1}(i, \text{dom},j)] = 0 \quad .
\end{aligned}$$

The third term on the RHS of (42.13) is

$$\text{Term}(3) = [1/(\text{COSTS}(j) - \text{TAX0}(j))] * \sum_{qq} \text{LABOCCIND}(qq,j) * x1laboi(qq,j) \quad . \quad (42.22)$$

Using $E_x1laboi$ [from subsection 18.8c], we obtain

$$\text{Term}(3) = [1/(\text{COSTS}(j) - \text{TAX0}(j))] * \text{LABIND_J}(j) * \text{labind}(j) \quad .$$

(42.23)

In deriving (42.23), we used the formula from subsection 18.5i that states that

$$\sum_{qq} \text{LABOCCIND}(qq, j) = \text{LABIND_J}(j) \quad .$$

In addition, we recognized that

$$\sum_{qq} \text{LABOCCIND}(qq, j) * a1laboi(qq, j) -$$

$$\sum_{qq} \text{LABOCCIND}(qq, j) \sum_m [\text{LABOCCIND}(m, j) / \text{LABIND_J}(j)] * a1laboi(m, j) = 0$$

and that⁷³

$$\begin{aligned} & \sum_{qq} \text{LABOCCIND}(qq, j) * \text{SIGMA1LABOCC}(qq, j) * \beta(qq, j) \\ & - [\sum_{qq} \text{LABOCCIND}(qq, j) * \text{SIGMA1LABOCC}(qq, j)] * \\ & \quad [\sum_m \text{SSTAR1LABOCC}(m, j) * \beta(m, j)] \\ & = \sum_{qq} \text{LABOCCIND}(qq, j) * \text{SIGMA1LABOCC}(qq, j) * \beta(qq, j) \\ & \quad - \sum_m \text{CRESHSUM2}(j) * \text{SSTAR1LABOCC}(m, j) * \beta(m, j) \\ & = \sum_{qq} \text{LABOCCIND}(qq, j) * \text{SIGMA1LABOCC}(qq, j) * \beta(qq, j) \\ & \quad - \sum_m \text{LABOCCIND}(m, j) * \text{SIGMA1LABOCC}(m, j) * \beta(m, j) = 0 , \end{aligned}$$

where $\beta(qq, j)$ can be interpreted as either $p1laboi(qq, j)$ or $a1laboi(qq, j)$.

Together, the third, fourth and fifth terms on the RHS of (42.13) can now be written as

$$\text{Terms}(3 \text{ to } 5) = [1/(\text{COSTS}(j) - \text{TAX0}(j))] *$$

$$\{\text{LABIND_J}(j) * \text{labind}(j) + \text{CAPITAL}(j) * \text{cap_at_t}(j) + \text{LAND}(j) * \text{n}(j)\} \quad . \quad (42.24)$$

Using E_labind , $E_cap_at_t$ and E_p1land [from subsection 18.8c], we obtain

$$\text{Terms}(3 \text{ to } 5) =$$

$$\begin{aligned} & [1/(\text{COSTS}(j) - \text{TAX0}(j))] * \{ \text{TOTFACIND}(j) * [z(j) + a1(j) + a1prim(j) + a1primgen] \\ & + \text{LABIND_J}(j) * [a1lab(j) + a1labgen] + \text{CAPITAL}(j) * a1cap(j) \} \end{aligned}$$

⁷³ It is helpful to note from subsection 18.5m that

$$\text{CRESHSUM2}(j) = \sum_{qq} \text{LABOCCIND}(qq, j) * \text{SIGMA1LABOCC}(qq, j)$$

and that $\text{SSTAR1LABOCC}(m, j) = \text{LABOCCIND}(m, j) * \text{SIGMA1LABOCC}(m, j) / \text{CRESHSUM2}(j)$.

$$+ \text{LAND}(j) * a_{1\text{land}}(j) \}. \quad (42.25)$$

In deriving (42.25) we used the formula from subsection 18.5i that states that

$$\text{TOTFACIND}(j) = \text{LABIND}_J(j) + \text{CAPITAL}(j) + \text{LAND}(j) \ .$$

In addition, we recognized that⁷⁴

$$\begin{aligned} & - \text{LABIND}_J(j) * \text{SIGMA1LAB}(j) * \alpha - \text{CAPITAL}(j) * \text{SIGMA1CAP}(j) * \beta \\ & - \text{LAND}(j) * \text{SIGMA1LND}(j) * \gamma \\ & + [\text{LABIND}_J(j) * \text{SIGMA1LAB}(j) + \text{CAPITAL}(j) * \text{SIGMA1CAP}(j) \\ & + \text{LAND}(j) * \text{SIGMA1LND}(j)] * [\text{SSTAR1FAC}(\text{labour}, j) * \alpha \\ & + \text{SSTAR1FAC}(\text{capital}, j) * \beta + \text{SSTAR1FAC}(\text{land}, j) * \gamma] = 0 \ , \quad (42.26) \end{aligned}$$

for any values of α , β and γ . In particular, (42.26) holds when

$$\alpha = p_{1\text{lab}}(j), \beta = p_{1\text{cap}}(j) \text{ and } \gamma = p_{1\text{land}}(j) \ ,$$

and when $\alpha = (a_{1\text{lab}}(j) + a_{1\text{labgen}})$, $\beta = a_{1\text{cap}}(j)$ and $\gamma = a_{1\text{land}}(j)$.

Finally, we recognized that⁷⁵

$$\begin{aligned} & \{ \text{LABIND}_J(j) * \text{SOURCE_SHRLK}(\text{capital}, j) \\ & - \text{CAPITAL}(j) * \text{SOURCE_SHRLK}(\text{labour}, j) \} * \text{twistlk}(j) = 0 \ . \quad (42.27) \end{aligned}$$

The last term on the RHS of (42.13) is

$$\text{Term}(6) = [1 / (\text{COSTS}(j) - \text{TAX0}(j))] * \text{OTHCOST}(j) * x_{1\text{oct}}(j) \ . \quad (42.28)$$

From $E_{x_{1\text{oct}}}$ [in subsection 18.8b], we obtain

$$\text{Term}(6) = [1 / (\text{COSTS}(j) - \text{TAX0}(j))] * \text{OTHCOST}(j) * [z(j) + a_{1(j)} + a_{1\text{oct}}(j)] \ . \quad (42.29)$$

In terms of the strategy outlined earlier for establishing E_z , (42.21), (42.25) and (42.29) combined corresponds to (42.6).

Using the expressions in (42.19), (42.21), (42.25) and (42.29) for Term(1), Term(2), Terms(3 to 5) and Term(6), we find that

$$\begin{aligned} \psi(j) = & - z(j) + a_{0\text{ind}}(j) + \sum_{cc} \text{H0CC}(cc, j) * a_{0\text{ccom}}(cc, j) \\ & + \sum_{i \in \text{COM_JP}} \text{H0CI}(i, j) * [a_{0\text{com}}(i) + a_{0\text{ci}}(i, j)] \end{aligned}$$

⁷⁴ In deriving (42.26), it is helpful to note from subsection 18.5m that $\text{LABIND}_J * \text{SIGMA1LAB}(j) + \text{CAPITAL}(j) * \text{SIGMA1CAP}(j) + \text{LAND}(j) * \text{SIGMA1LND}(j) = \text{CRESHSUM1}(j)$ and that $\text{SSTAR1FAC}(\text{labour}, j) = \text{LABIND}_J(j) * \text{SIGMA1LAB}(j) / \text{CRESHUM1}(j)$, $\text{SSTAR1FAC}(\text{capital}, j) = \text{CAPITAL}(j) * \text{SIGMA1CAP}(j) / \text{CRESHUM1}(j)$, and $\text{SSTAR1FAC}(\text{land}, j) = \text{LAND}(j) * \text{SIGMA1LND}(j) / \text{CRESHUM1}(j)$.

⁷⁵ In deriving (42.27), it is helpful to note the definition in subsection 18.5m of SOURCE_SHRLK .

$$\begin{aligned}
& + [1/(COSTS(j)-TAX0(j))] * \{ \sum_i TPURCHVAL1(i,j) * [z(j) + a1(j) + a1ci(i,j)] \\
& \quad + \sum_i \sum_s PURCHVAL1(i,s,j) * a1csi(i,s,j) \\
& \quad + \sum_{i \in TRANSERV} PURCHVAL1(i, imp, j) * fl_trans(j) \\
& \quad + \sum_{i \in COMMUNICAT} PURCHVAL1(i, imp,j) * fl_commun(j) \} \\
& + [1/(COSTS(j)-TAX0(j))] * \{ TOTFACIND(j) * \\
& \quad [z(j) + a1(j) + a1prim(j) + a1primgen] + LABIND_J(j) * [a1lab(j) + a1labgen] \\
& \quad + CAPITAL(j) * a1cap(j) + LAND(j) * a1land(j) \} \\
& + [1/(COSTS(j)-TAX0(j))] * OTHCOST(j) * [z(j) + a1(j) + a1oct(j)] , \\
& \hspace{15em} j \in IND_JP. \quad (42.30)
\end{aligned}$$

The RHS of (42.30) can be simplified using the formula in subsection 18.5j which implies that:

$$COSTS(j) - TAX0(j) = \sum_i TPURCHVAL1(i,j) + TOTFACIND(j) + OTHCOST(j) . \quad (42.31)$$

Applying (42.31) in (42.30) we find that the $z(j)$ terms cancel and that the coefficient on $a1(j)$ is one. Thus, we find that the RHS of (42.30) is the same as $a(j)$ given by E_a_JP . This corresponds in our strategy outlined earlier to establishing (42.7).

Having now demonstrated that $\psi(j) = a(j)$, that is having established (42.14), we have completed the proof of (I) and (II).

42.2. Derivation of E_z for unique-product industries

For unique-product industries ($j \in IND_UP$) the zero pure profits condition can be written as

$$\begin{aligned}
\sum_{i \in COM_UP} MAKE(i,j) & = \sum_i \sum_s PURCHVAL1(i,s,j) + LABIND_J(j) \\
& + CAPITAL(j) + LAND(j) + OTHCOST(j) + TAX0(j) , \quad j \in IND_UP, \quad (42.32)
\end{aligned}$$

which leads to:

$$\begin{aligned}
(2-POW_TAX0(j)) * \sum_{i \in COM_UP} P0DOM(i) * X0DOM(i) * H0CI(i,j) & = \\
\sum_{i \in COM} \sum_{s \in SOURCE} P1CSI(i,s,j) * X1CSI(i,s,j) & \\
+ \sum_{qq \in OCC} P1LABOI(qq,j) * X1LABOI(qq,j) & \\
+ P1CAP(j) * CAP_AT_T(j) + P1LAND(j) * N(j) & \\
+ P1OCT(j) * X1OCT(j) , & \hspace{10em} j \in IND_UP , \quad (42.33)
\end{aligned}$$

where $H0CI(i,j)$ is one if industry j produces commodity i and zero otherwise. A percentage-change form of (42.33) is:

$$\begin{aligned} \sum_{i \in \text{COM_UP}} H0CI(i,j) * p0\text{dom}(i) &= \phi(j) \\ &+ [1/(\text{COSTS}(j) - \text{TAX0}(j))] * \{ \sum_i \sum_s \text{PURCHVAL1}(i,s,j) * p1\text{csi}(i,s,j) \\ &+ \sum_{qq} \text{LABOCCIND}(qq,j) * p1\text{labo}(qq,j) + \text{CAPITAL}(j) * p1\text{cap}(j) \\ &+ \text{LAND}(j) * p1\text{land}(j) + 100 * \text{ROTHCOST}(j) * \text{del}_p1\text{oct}(j) \} \\ &+ \frac{\text{POW_TAX0}(j)}{(2 - \text{POW_TAX0}(j))} * \text{powtax0}(j), \quad j \in \text{IND_JP}, \end{aligned} \quad (42.34)$$

where

$$\begin{aligned} \phi(j) &= - \sum_{i \in \text{COM_UP}} H0CI(i,j) * x0\text{dom}(i) \\ &+ [1/(\text{COSTS}(j) - \text{TAX0}(j))] * \{ \sum_i \sum_s \text{PURCHVAL1}(i,s,j) * x1\text{csi}(i,s,j) \\ &+ \sum_{qq} \text{LABOCCIND}(qq,j) * x1\text{labo}(qq,j) + \text{CAPITAL}(j) * \text{cap_at_t}(j) \\ &+ \text{LAND}(j) * n(j) + \text{OTHCOST}(j) * x1\text{oct}(j) \} \quad \text{for } j \in \text{IND_UP}. \end{aligned} \quad (42.35)$$

Our aim in this subsection is to show that

$$M + E_z(j \in \text{IND_UP}) \Rightarrow (42.34) \text{ and } (42.35) \quad (\text{III})$$

and

$$M + (42.34) \text{ and } (42.35) \Rightarrow E_z(j \in \text{IND_UP}), \quad (\text{IV})$$

where M is, as before, the MONASH equations excluding E_z for all $j \in \text{IND}$. By establishing (III) and (IV) we will have shown that

$$M + E_z(j \in \text{IND_UP}) \Leftrightarrow M + (42.34) \text{ and } (42.35).$$

In other words, we will have shown in the context of the MONASH model that representing the zero-pure-profits conditions for unique-product industries as $E_z(j \in \text{IND_UP})$ is equivalent to representing them as (42.34) and (42.35).

On inspecting (42.34), $E_p0\text{ind}$ ⁷⁶ and E_z we see that both (III) and (IV) can be proved by showing that

$$M \Rightarrow \phi(j) = a(j) \quad \text{for all } j \in \text{IND_UP} \quad (42.36)$$

where $\phi(j)$ is defined by (42.35) and $a(j)$ is as defined in E_a_UP .

The first term on the RHS of (42.35) is

$$\text{Term}(1) = - \sum_{i \in \text{COM_UP}} H0CI(i,j) * x0\text{dom}(i) \quad . \quad (42.37)$$

⁷⁶ Recall that unique-product industries produce only one-industry products. Hence, for $j \in \text{IND_UP}$, $\sum_{i \in \text{COM_UP}} H0CI(i,j) * p0\text{dom}(i) = \sum_{i \in \text{COM}} H0CI(i,j) * p0\text{dom}(i)$.

From E_{x0dom_UP} [in subsection 18.8a], we find that

$$\begin{aligned}
 \text{Term}(1) &= - \sum_{i \in \text{COM_UP}} H0CI(i,j) \sum_{k \in \text{IND_UP}} B0CI(i,k) * [z(k) - a0ind(k) - a0com(i)] \\
 &= - \sum_{i \in \text{COM_UP}} \sum_{k \in \text{IND_UP}} H0CI(i,j) * B0CI(i,k) * [z(k) - a0ind(k) - a0com(i)] \\
 &= - \sum_{i \in \text{COM_UP}} H0CI(i,j) * [z(j) - a0ind(j) - a0com(i)] \\
 &= -z(j) + a0ind(j) + \sum_{i \in \text{COM_UP}} H0CI(i,j) * a0com(i) . \quad (42.38)
 \end{aligned}$$

For Terms 2 to 6 on the RHS of (42.35) we can derive identical expressions to (42.21), (42.25) and (42.29). Combining these expressions with (42.38) leads eventually to (42.36), and therefore to the completion of our derivation of (III) and (IV).

Together (I) – (IV) establish in the context of the other equations in MONASH that E_z is a legitimate representation of zero pure profits in production.

42.3. Derivation of the GDP identity both in levels and percentage changes

At the start of any MONASH computation (42.10) and (42.32) are satisfied, reflecting the first balance condition in our input-output data, see subsection 12.1. The previous two subsections show that the MONASH equations imply (42.12), (42.13), (42.34) and (42.35). These are percentage change versions of (42.10) and (42.32).⁷⁷ This means that (42.10) and (42.32) will be satisfied at every step of a multi-step MONASH computation.

By adding (42.10) over all $j \in \text{IND_JP}$ and (42.32) over all $j \in \text{IND_UP}$, then aggregating the results and drawing on various formulas especially in subsections 18.5c and i, we obtain:

$$\begin{aligned}
 \sum_{i \in \text{COM}} \text{MAKE_SALES}(i) &= \sum_j \sum_i \sum_s \text{PURCHVAL1}(i,s,j) \\
 &+ \text{AGGLAB} + \text{AGGCAP} + \text{AGGLND} + \text{AGGOCT} + \sum_j \text{TAX0}(j) . \quad (42.39)
 \end{aligned}$$

Our initial input output data satisfy:⁷⁸

$$\text{MAKE_SALES}(i) = \text{SALES}(i) , \quad i \in \text{COM} . \quad (42.40)$$

⁷⁷ Equations (42.12), (42.13), (42.34) and (42.35) are a percentage change version of (42.11) and (42.33). Can we be sure that they are a percentage change version of (42.10) and (42.32)? In other words, can we be sure that throughout MONASH computations: $\text{MAKE}(i,j)$ equals $\text{P0DOM} * \text{X0CI}(i,j)$; $\text{PURCHVAL1}(i,s,j)$ equals $\text{P1CSI}(i,s,j) * \text{X1CSI}(i,s,j)$; etc? As explained in subsection 12.2, by adopting simple conventions regarding quantity units, we can be sure that these relationships are satisfied in our initial solution, $\bar{V}(0)$, for year 0. Via update statements and formulas, they continue to be satisfied in subsequent solutions. For example, in subsection 18.7, we see that the movement in $\text{MAKE}(i,j)$ reflects the movements in $\text{P0DOM}(i)$ and $\text{X0CI}(i,j)$.

⁷⁸ This is the second balance condition, see subsection 12.1.

Via the formula for SALES(i) in subsection 18.5j, the market-clearing equations E_p0dom_NM and E_p0dom_M in subsection 18.8i, and the update statements for BAS1, BAS2, ..., MAR1, MAR2, ..., etc. in subsection 18.7, we see that changes in SALES(i) in multi-step MONASH solutions reflect the percentage changes in the basic price and quantity of domestic commodity i [p0dom(i) and x0dom(i)]. Similarly, via the formula for MAKE_SALES(i) in subsection 18.5c, E_x0dom_JP in subsection 18.8a and the update statements for MAKE(i,j) in subsection 18.7, we see that the changes in MAKE_SALES(i) in multi-step MONASH solutions also reflect p0dom(i) and x0dom(i). Thus, (42.40) will be maintained throughout multi-step MONASH solutions. Combining (42.40) with (42.39), we find that the equation

$$\begin{aligned} \sum_{i \in \text{COM}} \text{SALES}(i) &= \sum_j \sum_i \sum_s \text{PURCHVAL1}(i,s,j) \\ &+ \text{AGGLAB} + \text{AGGCAP} + \text{AGGLND} + \text{AGGOCT} + \sum_j \text{TAX0}(j) \end{aligned} \quad (42.41)$$

will be valid throughout multi-step solutions.

From (42.41), we can quickly move to (12.1) and then to (12.2). This implies that

$$\text{GDPEX} = \text{GDPIN} \quad (42.42)$$

not only in the initial database but throughout multi-step solutions.

Having established that GDPEX (defined by a formula in subsection 18.5h) is always equal to GDPIN (defined by a formula in subsection 18.5i), we can conclude that MONASH computations will satisfy

$$\text{gdpexp} = \text{gdpinc} \quad (42.43)$$

where

gdpexp is the percentage change in GDP from the expenditure side (calculated in E_gdpexp, E_xigdp and E_gdpreal, subsection 18.8l); and

gdpinc is the percentage change in GDP from the income side (calculated in E_gdpinc, subsection 18.8l).

42.4. Demonstration that gdprealinc equals gdpreal

Our strategy is as follows. First, we work with (42.13), (42.14), (42.35) and (42.36) to establish that

$$\begin{aligned} \sum_{j \in \text{IND}} (\text{COSTS}(j) - \text{TAX0}(j)) * a(j) &= - \sum_{i \in \text{COM}} \text{SALES}(i) * x0dom(i) \\ &+ \sum_{j \in \text{IND}} \text{TAX0}(j) * x0ind(j) \\ &+ \sum_{j \in \text{IND}} \sum_i \sum_s \text{PURCHVAL1}(i,s,j) * x1csi(i,s,j) + \text{Primfac} \quad , \end{aligned} \quad (42.44)$$

where

$$\begin{aligned}
\text{Primfac} &= \sum_{j \in \text{IND}} \sum_{qq} \text{LABOCCIND}(qq,j) * x1\text{labo}(qq,j) \\
&+ \sum_{j \in \text{IND}} \text{CAPITAL}(j) * \text{cap_at_t}(j) \\
&+ \sum_{j \in \text{IND}} \text{LAND}(j) * n(j) + \sum_{j \in \text{IND}} \text{OTHCOST}(j) * x1\text{oct}(j) . \quad (42.45)
\end{aligned}$$

Next we use market-clearing equations to replace $x0\text{dom}(i)$ in the first term on the RHS of (42.44) with percentage changes in demands for domestically produced good i . Intermediate input demands then cancel with the third term on the RHS, leaving percentage changes in final demands. After a considerable amount of algebra, these final demands are aggregated to form gdpreal [as defined in $E_gdpreal$]. Finally, after some rearrangements we establish that $\text{GDPEX} * \text{gdpreal}$ is given by the RHS of $E_gdprealinc$, thus demonstrating that gdprealinc is the same as gdpreal .

We start by noting that (42.13) and (42.14) give:

$$\begin{aligned}
&\sum_{j \in \text{IND_JP}} (\text{COSTS}(j) - \text{TAX0}(j)) * a(j) \\
&= - \sum_{j \in \text{IND_JP}} \sum_{i \in \text{COM_JP}} \text{COSTS}(j) * \text{H0CI}(i,j) * x0\text{ci}(i,j) \\
&+ \sum_{j \in \text{IND_JP}} \sum_{i \in \text{COM_JP}} \text{TAX0}(j) * \text{H0CI}(i,j) * x0\text{ci}(i,j) \\
&+ \sum_{j \in \text{IND_JP}} \sum_i \sum_s \text{PURCHVAL1}(i,s,j) * x1\text{csi}(i,s,j) + \text{Primfac_JP} , \quad (42.46)
\end{aligned}$$

where

Primfac_JP is given by the RHS of (42.45) except that the sums over j are restricted to $j \in \text{IND_JP}$.

Via (42.10) and the formula in subsection 18.5j for $\text{COSTS}(j)$, we see that

$$\text{COSTS}(j) = \sum_{i \in \text{COM_JP}} \text{MAKE}(i,j) , \quad j \in \text{IND_JP} . \quad (42.47)$$

Hence, via the formula in subsection 18.5c for $\text{MAKE_COSTS}(j)$, we can write

$$\text{COSTS}(j) = \text{MAKE_COSTS}(j) , \quad j \in \text{IND_JP} . \quad (42.48)$$

Then via the formula in subsection 18.5c for $\text{H0CI}(i,j)$, we see that (42.46) can be written as

$$\begin{aligned}
&\sum_{j \in \text{IND_JP}} (\text{COSTS}(j) - \text{TAX0}(j)) * a(j) \\
&= - \sum_{j \in \text{IND_JP}} \sum_{i \in \text{COM_JP}} \text{MAKE}(i,j) * x0\text{ci}(i,j) \\
&+ \sum_{j \in \text{IND_JP}} \text{TAX0}(j) * \sum_{i \in \text{COM_JP}} \{ \text{MAKE}(i,j) / \text{COSTS}(j) \} * x0\text{ci}(i,j) \\
&+ \sum_{j \in \text{IND_JP}} \sum_i \sum_s \text{PURCHVAL1}(i,s,j) * x1\text{csi}(i,s,j) + \text{Primfac_JP} . \quad (42.49)
\end{aligned}$$

By using the formula in subsection 18.5c for $B0CI(i,j)$ together with (42.40) and E_{x0dom_JP} in subsection 18.8a, we can simplify the first term on the RHS of (42.49), and by using (42.47) and E_{x0ind_JP} in subsection 18.8o, we can simplify the second term. These simplifications lead to

$$\begin{aligned} \sum_{j \in IND_JP} (COSTS(j) - TAX0(j)) * a(j) = & - \sum_{i \in COM_JP} SALES(i) * x0dom(i) \\ & + Primfac_JP + \sum_{j \in IND_JP} TAX0(j) * x0ind(j) \\ & + \sum_{j \in IND_JP} \sum_i \sum_s PURCHVAL1(i,s,j) * x1csi(i,s,j). \end{aligned} \quad (42.50)$$

Working with (42.35) and (42.36), we can derive an equation similar to (42.50) for $(COSTS(j) - TAX0(j)) * a(j)$ aggregated over all $j \in IND_UP$. This equation, together with (42.50), leads to (42.44) and (42.45).

By substituting into (42.44) from the market clearing conditions (E_{p0dom_NM} and E_{p0dom_M} in subsection 18.8i) and from the formula in subsection 18.5d for $PURCHVAL1(i,s,j)$, we find that

$$\begin{aligned} \sum_{j \in IND} (COSTS(j) - TAX0(j)) * a(j) = & - \sum_{r \in COM} \sum_j BAS1(r, dom, j) * x1csi(r, dom, j) \\ & - \sum_{r \in COM} \sum_j BAS2(r, dom, j) * x2csi(r, dom, j) - \sum_{r \in COM} BAS3(r, dom) * x3cs(r, dom) \\ & - \sum_{r \in COM} BAS4(r) * x4(r) - \sum_{r \in COM} BAS5(r, dom) * x5cs(r, dom) \\ & - 100 * \sum_{r \in COM} P0LEV(r, dom) * d_x6cs(r, dom) \\ & - \sum_{r \in MARGCOM} \sum_i \sum_s \sum_j MAR1(i,s,j,r) * x1marg(i,s,j,r) \\ & - \sum_{r \in MARGCOM} \sum_i \sum_s \sum_j MAR2(i,s,j,r) * x2marg(i,s,j,r) \\ & - \sum_{r \in MARGCOM} \sum_i \sum_s MAR3(i,s,r) * x3marg(i,s,r) \\ & - \sum_{r \in MARGCOM} \sum_i MAR4(i,r) * x4marg(i,r) \\ & - \sum_{r \in MARGCOM} \sum_i \sum_s MAR5(i,s,r) * x5marg(i,s,r) + \sum_{j \in IND_JP} TAX0(j) * x0ind(j) \\ & + \sum_{r \in COM} \sum_j \sum_s [BAS1(r,s,j) + TAX1(r,s,j) \\ & + \sum_{k \in MARGCOM} MAR1(r,s,j,k)] * x1csi(r,s,j) + Primfac \quad . \end{aligned} \quad (42.51)$$

By continuing to use formulas from subsection 18.5d for purchasers' values, by substituting from E_{x1marg} , ..., E_{x5marg} and by cancelling some $BAS1$ and $MAR1$ terms, we can rewrite (42.51) as

$$\sum_j (COSTS(j) - TAX0(j)) * a(j) = - \sum_{r \in COM} \sum_s \sum_j PURCHVAL2(r,s,j) * x2csi(r,s,j)$$

$$\begin{aligned}
& - \sum_{r \in \text{COM}} \sum_s \text{PURCHVAL3}(r,s) * x3cs(r,s) - \sum_{r \in \text{COM}} \text{PURCHVAL4}(r) * x4(r) \\
& - \sum_{r \in \text{COM}} \text{PURCHVAL5}(r,s) * x5cs(r,s) - 100 * \sum_{r \in \text{COM}} \sum_s \text{POLEV}(r,s) * d_x6cs(r,s) \\
& - \text{Techchangemargins} + \text{Realtaxes} + \text{Realimportsbasic} + \text{Primfac} \quad , \quad (42.52)
\end{aligned}$$

where

$$\begin{aligned}
\text{Techchangemargins} = & \sum_{r \in \text{MARGCOM}} \sum_i \sum_s \sum_j \text{MAR1}(i,s,j,r) * a1\text{marg}(i,s,j,r) \\
& + \sum_{r \in \text{MARGCOM}} \sum_i \sum_s \sum_j \text{MAR2}(i,s,j,r) * a2\text{marg}(i,s,j,r) \\
& + \sum_{r \in \text{MARGCOM}} \sum_i \sum_s \text{MAR3}(i,s,r) * a3\text{marg}(i,s,r) \\
& + \sum_{r \in \text{MARGCOM}} \sum_i \text{MAR4}(i,r) * a4\text{marg}(i,r) \\
& + \sum_{r \in \text{MARGCOM}} \sum_i \sum_s \text{MAR5}(i,s,r) * a5\text{marg}(i,s,r) \quad , \quad (42.53)
\end{aligned}$$

$$\begin{aligned}
\text{Realtaxes} = & \sum_{j \in \text{IND}} \text{TAX0}(j) * x0\text{ind}(j) + \sum_{r \in \text{COM}} \sum_s \sum_j \text{TAX1}(r,s,j) * x1\text{csi}(r,s,j) \\
& + \sum_{r \in \text{COM}} \sum_s \sum_j \text{TAX2}(r,s,j) * x2\text{csi}(r,s,j) + \sum_{r \in \text{COM}} \sum_s \text{TAX3}(r,s,j) * x3\text{cs}(r,s) \\
& + \sum_{r \in \text{COM}} \text{TAX4}(r) * x4(r) + \sum_{r \in \text{COM}} \text{TAX5}(r,s) * x5s(r,s) \quad , \quad \text{and} \quad (42.54)
\end{aligned}$$

$$\begin{aligned}
\text{Realimportsbasic} = & \sum_{r \in \text{COM}} \text{BAS1}(r, \text{imp}, j) * x1\text{csi}(r, \text{imp}, j) \\
& + \sum_{r \in \text{COM}} \text{BAS2}(r, \text{imp}, j) * x2\text{csi}(r, \text{imp}, j) + \sum_{r \in \text{COM}} \text{BAS3}(r, \text{imp}) * x3\text{cs}(r, \text{imp}) \\
& + \sum_{r \in \text{COM}} \text{BAS5}(r, \text{imp}) * x5\text{cs}(r, \text{imp}) \\
& + 100 * \sum_{r \in \text{COM}} \text{POLEV}(r, \text{imp}) * d_x6cs(r, \text{imp}) \quad . \quad (42.55)
\end{aligned}$$

Next, we relate each of the first five terms on the RHS of (42.52) to the real expenditure aggregates appearing on the RHS of $E_gd\text{preal}$.

Via $E_x2\text{csi}$ in subsection 18.8d, we find that the first term is given by:

$$\text{Term}(1) = - \sum_r \sum_s \sum_j \text{PURCHVAL2}(r,s,j) * [y(j) + a2\text{ind}(j) + a2\text{ci}(r,j) + a2\text{csi}(r,s,j)] \quad . \quad (42.56)$$

In deriving (42.56), we recognized that when $E_x2\text{csi}$ is multiplied through by $\text{PURCHVAL2}(i,s,j)$ and aggregated over s , then the price, relative technology and twist terms drop out. Using the formulas in subsections 18.5d and 18.5h for $\text{TPURCHVAL2}(r,j)$ and $\text{VINVEST}(j)$, together with $E_r_inv_cap_u$ [in subsection 18.8l], we see that

$$\text{Term}(1) = - \text{AGGINV} * \text{ir} - \text{Techchangeinv} \quad , \quad (42.57)$$

where

$$\text{Techchangeinv} = \sum_r \sum_s \sum_j \text{PURCHVAL2}(r,s,j) * \{ a2\text{ind}(j) + a2\text{ci}(r,j) + a2\text{csi}(r,s,j) \} \quad (42.58)$$

Via E_{x3cs} [in subsection 18.8e] and the formula for $\text{TPURCHVAL3}(r,s)$ in subsection 18.5d, we find that the second term on the RHS of (42.52) is given by

$$\text{Term}(2) = - \sum_r \text{TPURCHVAL3}(r) * x3(r) \quad . \quad (42.59)$$

Then via E_{x3} [in subsection 18.8e] and the formulas in subsections 18.5h and 18.5k for AGGCON and $\text{S3COM}(r)$, we obtain

$$\begin{aligned} \text{Term}(2) = & - \text{AGGCON} * \{ q + \sum_r \text{S3COM}(r) * \text{EPS}(r) * (c-q) \\ & + \sum_r \sum_j \text{S3COM}(r) * \text{ETA}(r,j) * p3(j) + \sum_r \text{S3COM}(r) * [a3\text{com}(r) - \text{ave_a3com}] \} \quad . \end{aligned} \quad (42.60)$$

By using $E_{\text{ave_a3com}}$ and by recalling from section 22 [see (22.9) and (22.10)] that

$$\sum_r \text{S3COM}(r) * \text{ETA}(r,j) = - \text{S3COM}(j) \text{ and } \sum_r \text{EPS}(r) * \text{S3COM}(r) = 1 \quad ,$$

we can simplify (42.60) to

$$\text{Term}(2) = - \text{AGGCON} * [c - \sum_j \text{S3COM}(j) * p3(j)] \quad .$$

Hence, via E_{p3} in subsection 18.8e and E_{phi} and E_c in subsection 18.8l,

$$\text{Term}(2) = - \text{AGGCON} * [c - x_i3] = - \text{AGGCON} * c_r \quad . \quad (42.61)$$

The third, fourth and fifth terms on the RHS of (42.52) can be written as:

$$\text{Term}(3) = - \text{AGGEXP} * \text{expvol}; \quad (42.62)$$

$$\text{Term}(4) = - \text{AGGOTH} * \text{othreal}; \quad \text{and} \quad (42.63)$$

$$\text{Term}(5) = - 100 * d_{\text{inventreal}} \quad (42.64)$$

(see E_{expvol} , E_{f5gen} and $E_{d_{\text{inventreal}}}$ in subsection 18.8l).

Now we return to (42.55) and relate Realimportsbasic to the import term on the RHS of E_{gdpreal} . From (42.55) and E_{x0imp} [in subsection 18.8i] we find that

$$\text{Realimportsbasic} = \sum_r \text{IMPORTS}(r) * x0\text{imp}(r) \quad . \quad (42.65)$$

Then by using the formula in subsection 18.5f for $\text{IMPCOST}(r)$, we obtain

$$\text{Realimportsbasic} = \sum_r \text{IMPCOST}(r) * x0\text{imp}(r) + \text{Realtariffs} \quad (42.66)$$

where

$$\text{Realtariffs} = \sum_r \text{TARIFF}(r) * x0\text{imp}(r) . \quad (42.67)$$

Hence,

$$\text{Realimportsbasic} = \text{AGGIMP} * \text{impvol} + \text{Realtariffs} , \quad (42.68)$$

(see E_impvol in subsection 18.8I).

Finally, by substituting from (42.57), (42.61), (42.62), (42.63), (42.64) and (42.68) into (42.52), we see that

$$\begin{aligned} \sum_j (\text{COSTS}(j) - \text{TAX0}(j)) * a(j) = & - \text{AGGINV} * ir - \text{AGGCON} * cr - \text{AGGEXP} * \text{expvol} \\ & - \text{AGGOTH} * \text{othreal} - 100 * d_inventreal - \text{Techchangeinv} - \text{Techchangemargins} \\ & + \text{Realtaxes} + \text{AGGIMP} * \text{impvol} + \text{Realtariffs} + \text{Primfac} . \end{aligned} \quad (42.69)$$

On rearranging (42.69) and using $E_gdpreal$ [in subsection 18.8I] we obtain

$$\text{GDPEX} * \text{gdpreal} =$$

$$\begin{aligned} & - \sum_j (\text{COSTS}(j) - \text{TAX0}(j)) * a(j) - \text{Techchangeinv} - \text{Techchangemargins} \\ & + \text{Realtaxes} + \text{Realtariffs} + \text{Primfac} . \end{aligned} \quad (42.70)$$

On comparing the definitions of Techchangeinv , Techchangemargins , Realtaxes , Realtariffs and Primfac given in (42.58), (42.53), (42.54), (42.67) and (42.45) with terms on the RHS of $E_gdprealinc$ in subsection 18.8I, we conclude that

$$\text{GDPEX} * \text{gdpreal} = \text{GDPIN} * \text{gdprealinc} . \quad (42.71)$$

Via (42.42), (42.71) implies that

$$\text{gdpreal} = \text{gdprealinc} . \quad (42.72)$$

43. Appendix: the levels representation of price and quantity indexes in MONASH

All of the deviation (percentage change and change) equations used in our Johansen/Euler solutions of the MONASH model are listed in subsection 18.8. While many of the underlying levels equations are discussed in sections 19 to 36, we do not provide a complete listing of them. The forms of most of the levels equations are obvious from the corresponding deviation equations. However, this is not true for the equations defining percentage movements in price and quantity indexes (p and q).

In subsection 18.8 (see particularly 18.8I), these equations take the Divisia form:

$$p = \sum_k S_k p_k \quad (43.1)$$

and

$$q = \sum_k S_k q_k \quad (43.2)$$

where

p_k is the percentage change in the price of object (e.g., good) k ;

q_k is the percentage change in the quantity of k purchased; and

S_k is the share of k in the relevant expenditure aggregate.

The question to be considered in this appendix is the following: what are the levels equations which justify the use of deviation equations such as (43.1) and (43.2)?

It will be useful at the outset to provide an intuitive answer. We concentrate on price indexes. Given the level of a price index in year t , denoted by $P(t)$,⁷⁹ we define its level in year $t+1$ as the result obtained by: (a) moving the exogenous variables in small steps along a straight line⁸⁰ from their values in year t to their values in year $t+1$; (b) evaluating at each step along the way the percentage change, p , given by (43.1); and (c) generating the value of $P(t+1)$ by accumulating these percentage changes. By describing this process mathematically we obtain a levels version of (43.1). Usually, under this straight-line definition, P is close to the geometric average of the values of the relevant Laspeyres and Paasche indexes.

Three implications of our definition of P are worth noting. First, the Johansen/Euler method will produce legitimate solutions for P if and only if we implement it with a straight-line movement in the exogenous variables from their values in year t to their values in year $t+1$. If we adopted some other path, then accumulation of percentage changes computed via (43.1) would not necessarily produce a value equal to the value obtained by the straight-line movement required by our definition. For example, if exogenous taste-changes cause a reduction in the tobacco share of consumption and exogenous tax increases cause an increase in the tobacco price, then accumulation of percentage changes generated by (43.1) in a Johansen/Euler computation which introduced the taste changes followed by the tax increases would underestimate P . This is because if we introduced the taste changes first, then the tax-induced price change would receive a relatively low weight. Being restricted to straight-line movements (i.e. simultaneous taste and tax shocks representing equal percentages of their total shocks) is not a problem. This is the most obvious way to implement the Johansen/Euler method.

The second noteworthy implication of our definition of P relates to historical and decomposition simulations in which we use a straight-line Johansen/Euler method to compute the effects of movements in the exogenous variables from their values in year t to their values in year $t+\tau$, $\tau > 1$, without reference to years $t+1$, $t+2$, ..., $t+\tau-1$. In these simulations, accumulation of percentage changes from (43.1) produces a legitimate value for P only if the actual movements in the exogenous variables followed a straight line between years t and $t+\tau$, not just between adjacent

⁷⁹ We can assume that $P(0)$ equals one.

⁸⁰ This is defined precisely in (43.5).

years. Thus for example, in our historical simulation for 1987 to 1994 we assumed implicitly that the 34 per cent increase in the CPI that took place over this period was generated by straight-line changes in the exogenous variables.

The third implication, which is closely related to the second, concerns path dependence. In year-to-year simulations MONASH correctly generates the paths of Divisia indexes by moving the exogenous variables on a straight-line path from year 0 to year 1, and then on a different straight-line path from year 1 to year 2, etc. By the time we reach year T, the exogenous variables have moved on a non-linear path composed of T joined linear segments. The values of Divisia indexes in year T depend on these paths, not just their end points. Thus, in understanding the simulated change in the level of a Divisia index between year 0 and year T we often need to refer to movements in prices, quantities and shares in intermediate years.

The remainder of this section is organized as follows. Subsection 43.1 is a brief restatement of material from section 11 on the nature of the levels and deviations versions of MONASH and on the Johansen/Euler method. This facilitates the analysis in subsection 43.2 where we provide a formal mathematical statement of the levels equations which justify the use of (43.1) and (43.2). An interpretation of these levels equations in terms of Laspeyres and Paasche indexes is given in subsection 43.3.

43.1. Recapitulation: the levels and deviations versions of MONASH, and solution by numerical integration

In section 11, we asserted that the deviations version of MONASH (subsection 18.8) can be derived from a levels version of the form

$$F(\bar{V}) = 0 \quad . \quad (43.3)$$

The deviation version is a linear system of the form

$$A(\bar{V}) v = 0 \quad , \quad (43.4)$$

where

\bar{V} is a vector of values of the model's variables, either a solution [$F(\bar{V})=0$] or an approximate solution generated in a numerical integration using (43.4);

v is the vector of deviations⁸¹ in the values of the variables away from \bar{V} ; and

$A(\bar{V})$ is a matrix of partial derivatives of F [defined more precisely in (11.15) and (11.16)].

As explained in section 11, we solve MONASH by numerical integration (e.g., the Johansen/Euler method) using (43.4). In computing solutions for any year t , we

⁸¹ These can be either differences or percentage differences.

always have an initial solution V_I . New solutions are computed with the exogenous variables (X) set at points along the straight-line path

$$X(\alpha, X_I, X_F) = X_I + \alpha(X_F - X_I) \quad \text{for } \alpha \in [0, 1] \quad (43.5)$$

where

X_I gives the initial values of the exogenous variables (i.e., their values in V_I);
and

X_F is the required values of the exogenous variables.

In generating these new solutions, we start at $\alpha = 0$ (i.e., we start at X_I). We then increase the value of α and use (43.4) to evaluate the effects on the endogenous variables. Next, we update our solution, i.e., we move V from V_I to $V_I + dV$ where dV is the vector of effects caused by our movement in α . By making further movements in α , further evaluations of effects via (43.4) and further updates, we eventually derive the required solution, reflecting the required values for the exogenous variables (i.e., their values at $\alpha=1$). If the movements that we have made in α along the path from 0 to 1 are sufficiently small, then we asserted that our final computation will produce an accurate solution to (43.3) with the exogenous variables set at X_F .

A necessary condition for the validity of this assertion is the existence of the levels equations (43.3). Thus the question being considered in this appendix is significant.

43.2. The mathematics of the levels representations of the MONASH price and quantity indexes

In this subsection, we show that levels representations in (43.3) of price and quantity indexes (P and Q), compatible with the deviations equations (43.1) and (43.2), are as follows:

$$P = P_I \exp \int_{\alpha=0}^G 0.01 * [\Sigma_i \tilde{p}_i(\alpha, X_I, X_F) \tilde{S}_i(\alpha, X_I, X_F)] d\alpha \quad (43.6)$$

and

$$Q = Q_I \exp \int_{\alpha=0}^G 0.01 * [\Sigma_i \tilde{q}_i(\alpha, X_I, X_F) \tilde{S}_i(\alpha, X_I, X_F)] d\alpha \quad (43.7)$$

In these equations: G is an exogenous variable which has the value 0 in the initial solution (V_I) and 1 in the required solution (V_F); P_I and Q_I are the initial values of the price and quantity indexes; \tilde{p}_i , \tilde{q}_i and \tilde{S}_i are the percentage derivative in the price of i (P_i), the percentage derivative in the quantity of i (Q_i) and the expenditure share of i generated on paths (to be explained shortly) created by the movement of the exogenous variables (excluding G) between X_I and X_F ; and α is the fraction of the distance moved along the path from X_I to X_F .

Equations (43.6) and (43.7) are expressions for Divisia indexes.⁸² Before they can be understood, we need to define the paths along which \tilde{p}_i , \tilde{q}_i and \tilde{S}_i are generated. This definition should rely on computations in a system of equations which does not include (43.6) and (43.7). Otherwise we would be guilty of circular reasoning.

The system we use in our definitions of \tilde{p}_i , \tilde{q}_i and \tilde{S}_i consists of two parts. The first is the deviations system (43.4), i.e., the TABLO equations in subsection 18.8, including such equations as (43.1) and (43.2). The second is the set of update rules in subsection 18.7.⁸³

We refer to (43.4) together with the update rules as system A. This system is sufficient for us to define $P_i^A(\alpha^*, X_I, X_F)$, $Q_i^A(\alpha^*, X_I, X_F)$, $P^A(\alpha^*, X_I, X_F)$ and $Q^A(\alpha^*, X_I, X_F)$ as the values reached in system A by P_i , Q_i , P and Q after the exogenous variables (X) have travelled along the path defined by (43.5) with α moving in infinitesimally small steps from 0 to α^* .

With paths for P_i and Q_i defined by reference to system A, we define \tilde{p}_i , \tilde{q}_i and \tilde{S}_i by

$$\tilde{p}_i(\alpha, X_I, X_F) = [100/P_i^A(\alpha, X_I, X_F)] \partial P_i^A(\alpha, X_I, X_F) / \partial \alpha, \quad (43.8)$$

$$\tilde{q}_i(\alpha, X_I, X_F) = [100/Q_i^A(\alpha, X_I, X_F)] \partial Q_i^A(\alpha, X_I, X_F) / \partial \alpha \quad (43.9)$$

and

$$\tilde{S}_i(\alpha, X_I, X_F) = P_i^A(\alpha, X_I, X_F) * Q_i^A(\alpha, X_I, X_F) / \sum_j P_j^A(\alpha, X_I, X_F) * Q_j^A(\alpha, X_I, X_F). \quad (43.10)$$

Equations (43.8)-(43.10) complete the definitions of the RHSs of (43.6) and (43.7).

Now assume that (43.6) and (43.7) are part of the levels representation of our model. Our task is to show that the use of equations such as (43.1) and (43.2) in the deviations version is legitimate, i.e., we need to show that numerical integration using (43.4), which includes (43.1) and (43.2), will produce solutions to the levels version of the model, which includes (43.6) and (43.7).

In completing this task, the first point to recognize is that in deviation versions of (43.6) and (43.7), we should treat X_I and X_F as constants. In deviation versions

⁸² In learning about Divisia indexes, we found the conference volume edited by Eichhorn *et al.* (1978) to be helpful, especially the contribution by Vogt (1978).

⁸³ We assume that subsection 18.7 has been expanded to include update rules for all price, quantity and index variables. As presented, subsection 18.7 contains update rules for some of these variables, e.g. the CPI, but not for others. For the purpose of this appendix, a complete set of update rules is necessary because we need, in theory, to trace the paths of P_i , Q_i , P and Q as we move the exogenous variables from X_I to X_F .

of our model, we are concerned with effects on endogenous variables as we move exogenous variables along a path from X_I to X_F , not with movements in either X_I or X_F . Thus, with (43.6) and (43.7) forming part of the levels version of our model, equations suitable for inclusion in the deviation version are

$100*\text{del}P =$

$$P_I \left\{ \exp \int_{\alpha=0}^G 0.01 * [\Sigma_i \tilde{p}_i (\alpha, X_I, X_F) \tilde{S}_i (\alpha, X_I, X_F)] d\alpha \right\} * \Sigma_i \tilde{p}_i (G, X_I, X_F) \tilde{S}_i (G, X_I, X_F) \text{del}G \quad (43.11)$$

and

$$100*\text{del}Q = Q_I \left\{ \exp \int_{\alpha=0}^G 0.01 * [\Sigma_i \tilde{q}_i (\alpha, X_I, X_F) \tilde{S}_i (\alpha, X_I, X_F)] d\alpha \right\} * \Sigma_i \tilde{q}_i (G, X_I, X_F) \tilde{S}_i (G, X_I, X_F) \text{del}G \quad (43.12)$$

These equations, together with (43.6) and (43.7), give

$$p = \Sigma_i \tilde{p}_i (G, X_I, X_F) \tilde{S}_i (G, X_I, X_F) \text{del}G \quad (43.13)$$

and

$$q = \Sigma_i \tilde{q}_i (G, X_I, X_F) \tilde{S}_i (G, X_I, X_F) \text{del}G \quad (43.14)$$

where p and q are percentage changes in P and Q .

Next, we assume that if we replace equations such as (43.1) and (43.2) in (43.4) with equations such as (43.13) and (43.14), then the resulting system is a legitimate deviation version of our model. That is, we assume that if (\bar{G}, \bar{V}) is a solution of the levels version of the model, then deviations from that solution caused by small movements in G and in the other exogenous variables (X) can be calculated in the system:

$$\left[\begin{array}{c|c|c} A^*(\bar{V}) & A_{pq}^*(\bar{V}) & 0 \\ \hline 0 \dots 0 & 1 & 0 \\ \hline 0 \dots 0 & 0 & 1 \end{array} \middle| \begin{array}{c} -A_p(\bar{G}, X_I, X_F) \\ -A_q(\bar{G}, X_I, X_F) \end{array} \right] \begin{bmatrix} v^* \\ p \\ q \\ \text{del}G \end{bmatrix} = 0, \quad (43.15)$$

where

$A^*(\bar{V})$ is the $A(\bar{V})$ matrix in (43.4) excluding the rows for equations (43.1) and (43.2) and the columns for p and q ;

$A_{pq}^*(\bar{V})$ are the columns of $A(\bar{V})$ for p and q , excluding the rows for (43.1) and (43.2);

v^* is the vector of deviations in the variables excluding the deviations in P, Q and G;

$$A_p(\bar{G}, X_I, X_F) = \sum_i \tilde{p}_i(\bar{G}, X_I, X_F) \tilde{S}_i(\bar{G}, X_I, X_F) \quad ,$$

$$\text{and } A_q(\bar{G}, X_I, X_F) = \sum_i \tilde{q}_i(\bar{G}, X_I, X_F) \tilde{S}_i(\bar{G}, X_I, X_F) \quad .$$

In similar notation, (43.4) can be written as

$$\left[\begin{array}{c|cc} A^*(\bar{V}) & A_{pq}^*(\bar{V}) & \\ \hline -S^p(\bar{V}) & 1 & 0 \\ -S^q(\bar{V}) & 0 & 1 \end{array} \right] \begin{bmatrix} v^* \\ p \\ q \end{bmatrix} = 0 \quad , \quad (43.4')$$

where $S^p(\bar{V})$ and $S^q(\bar{V})$ are vectors of zeros and expenditure shares such that

$$S^p(\bar{V})v^* = \sum_i S_i(\bar{V}) p_i \quad \text{and} \quad S^q(\bar{V})v^* = \sum_i S_i(\bar{V}) q_i \quad .$$

Equation (43.15) is a valid deviations version of our model for calculating the effects of small changes in G and in the other exogenous variables along *any* path from their initial to their final values. However, we do not need this generality. We can choose to work only with the straight-line path, that is, we can restrict the use of (43.15) to computing the effects of small movements $\text{del}G$, in G combined with changes in the other exogenous variables from $X(G, X_I, X_F)$ to $X(G+\text{del}G, X_I, X_F)$.⁸⁴

With the scope of the movements in the exogenous variables restricted in this way, we can replace (43.15) in our computations with (43.4'). That is, rather than using (43.15) in a sequence of computations concerned with the effects of increases in G of $\text{del}G$ and changes in X from $X(G, X_I, X_F)$ to $X(G+\text{del}G, X_I, X_F)$, we can instead use (43.4') in a sequence of computations concerned only with the effects of movements in X from $X(G, X_I, X_F)$ to $X(G+\text{del}G, X_I, X_F)$.

The key to understanding why we can solve our model by a sequence of solutions of (43.4') rather than by a sequence of solutions of (43.15) is to recognize that with small values of $\text{del}G$, the two sequences of solutions produce the same paths for P and Q. In the sequence of solutions using (43.15), the percentage movement in P, for example, as we increase G by $\text{del}G$ and move X from $X(G, X_I, X_F)$ to $X(G+\text{del}G, X_I, X_F)$ is given by

$$p[43.15] = \sum_i \tilde{S}_i(G, X_I, X_F) \tilde{p}_i(G, X_I, X_F) \text{del}G \quad . \quad (43.16)$$

In view of the definition of $\tilde{p}_i(G, X_I, X_F)$,

$$\tilde{p}_i(G, X_I, X_F) \text{del}G = p_i[43.4'] \quad ,$$

⁸⁴ Recall from (43.5) that $X(G, X_I, X_F) = X_I + G^*(X_F - X_I)$.

where

$p_i[43.4']$ is the percentage effect on P_i , calculated in (43.4'), of moving X from $X(G, X_I, X_F)$ to $X(G+\text{del}G, X_I, X_F)$ as part of a sequence of small movements in the exogenous variables along the straight-line path from X_I to X_F .

Equation (43.16) can, therefore, be rewritten as

$$p[43.15] = \sum_i \tilde{S}_i(G, X_I, X_F) p_i[43.4'] \quad (43.17)$$

Hence, if the movements in G are small, then $p[43.15]$, calculated in (43.15) as the percentage effect on P of moving G from G to $G+\text{del}G$ and X from $X(G, X_I, X_F)$ to $X(G+\text{del}G, X_I, X_F)$, is the same as $p[43.4']$, calculated in (43.4') as the percentage effect on P of moving X from $X(G, X_I, X_F)$ to $X(G+\text{del}G, X_I, X_F)$. Thus, for small movements in G , the paths for P , and similarly for Q , are the same in calculations carried out with (43.15) and (43.4').

If the two sequences of solutions produce the same paths for P and Q , then they must also produce the same paths for the other variables. This is because in both systems, v^* satisfies

$$A^*(\bar{V})v^* + A_{pq}^*(\bar{V}) \begin{bmatrix} p \\ q \end{bmatrix} = 0 \quad (43.18)$$

With identical movements in P and Q , and in the exogenous components of v^* , the movements in the endogenous components of v^* will also be identical.

43.3. Interpretation of the MONASH price and quantity indexes

The most common indicators of aggregate price and quantity movements are Laspeyres and Paasche indexes. Laspeyres indexes use initial-situation weights. Expressed in notation similar to that appearing earlier in this section, Laspeyres indexes in multiplicative form are:

$$P(\text{Las}, X_F)/P(\text{Las}, X_I) = \prod_k [P_k(X_F)/P_k(X_I)] \tilde{S}_k^{(0, X_I, X_F)} \quad (43.19)$$

and

$$Q(\text{Las}, X_F)/Q(\text{Las}, X_I) = \prod_k [Q_k(X_F)/Q_k(X_I)] \tilde{S}_k^{(0, X_I, X_F)} \quad (43.20)$$

where

$P_k(X_I)$ and $P_k(X_F)$ are the initial and final prices of k , i.e., the prices with the exogenous variables at X_I and X_F ;

$Q_k(X_I)$ and $Q_k(X_F)$ are the initial and final quantities of k ;

$P(\text{Las}, X_I)$, $P(\text{Las}, X_F)$, $Q(\text{Las}, X_I)$ and $Q(\text{Las}, X_F)$ are the initial and final values of the Laspeyres price and quantity indexes, $P(\text{Las})$ and $Q(\text{Las})$; and

$\tilde{S}_k(\alpha, X_I, X_F)$ is the k th expenditure share when the exogenous variables are at $X_I + \alpha(X_F - X_I)$.

Paasche indexes use final-situation weights. They can be expressed as:

$$P(\text{Paa}, X_F)/P(\text{Paa}, X_I) = \prod_k [P_k(X_F)/P_k(X_I)] \tilde{S}_k(1, X_I, X_F) \quad (43.21)$$

and

$$Q(\text{Paa}, X_F)/Q(\text{Paa}, X_I) = \prod_k [Q_k(X_F)/Q_k(X_I)] \tilde{S}_k(1, X_I, X_F) . \quad (43.22)$$

In the Divisia indexes [(43.6) and (43.7)] used in MONASH, the percentage changes in prices and quantities, are broken into many parts of the form $p_i(G, X_I, X_F) \cdot \text{del}G$ and $q_i(G, X_I, X_F) \cdot \text{del}G$ where these are the percentage changes in the price and quantity of i arising from the change in the exogenous variables from $X_I + G \cdot (X_F - X_I)$ to $X_I + (G + \text{del}G) \cdot (X_F - X_I)$. In the calculation of P and Q , each of these parts is given a weight reflecting expenditure shares at the current value of the exogenous variables. For example, the percentage changes $p_i(G, X_I, X_F) \cdot \text{del}G$ and $q_i(G, X_I, X_F) \cdot \text{del}G$ are given the weight $\tilde{S}_i(G, X_I, X_F)$.

Because the weights applied to the percentage changes in P_i and Q_i move from $\tilde{S}_i(0, X_I, X_F)$ to $\tilde{S}_i(1, X_I, X_F)$, we expect the MONASH Divisia indexes to be given approximately by

$$P(\text{Div}, X_F)/P(\text{Div}, X_I) \simeq [P(\text{Las}, X_F)/P(\text{Las}, X_I)]^{1/2} [P(\text{Paa}, X_F)/P(\text{Paa}, X_I)]^{1/2} \quad (43.23)$$

and

$$Q(\text{Div}, X_F)/Q(\text{Div}, X_I) \simeq [Q(\text{Las}, X_F)/Q(\text{Las}, X_I)]^{1/2} [Q(\text{Paa}, X_F)/Q(\text{Paa}, X_I)]^{1/2} . \quad (43.24)$$

In one leading special case, (43.23) and (43.24) are exact. This is when:

- (i) the rates of change in the logarithms of P_k and Q_k with respect to α are constant [i.e., \tilde{p}_k and \tilde{q}_k defined in (43.8) and (43.9) do not change as we move α from 0 to 1]; and
- (ii) the expenditure shares adjust along straight lines according to

$$\tilde{S}_k(\alpha, X_I, X_F) = \tilde{S}_k(0, X_I, X_F) + \alpha [\tilde{S}_k(1, X_I, X_F) - \tilde{S}_k(0, X_I, X_F)] \quad \text{for all } k \text{ and for } \alpha \in [0, 1] . \quad (43.25)$$

Under these two conditions, (43.6) for example, gives⁸⁵

$$P(\text{Div}, X_F)/P(\text{Div}, X_I) = \exp \int_{\alpha=0}^1 \left\{ \sum_i \left(\frac{d \ln P_i^A}{d\alpha} \right) [\tilde{S}_i(0) + \alpha(\tilde{S}_i(1) - \tilde{S}_i(0))] \right\} d\alpha \quad (43.26)$$

⁸⁵ In (43.26) we simplify the notation slightly by omitting X_I and X_F as arguments of \tilde{S}_i .

Condition (i) implies that
$$\frac{d \ln P_i^A}{d\alpha} = \frac{\Delta \ln P_i^A}{\Delta\alpha} = \ln[P_i(X_F)/P_i(X_I)] .$$

Thus, (43.26) simplifies to

$$\begin{aligned} P(\text{Div}, X_F)/P(\text{Div}, X_I) &= \exp \sum_i \ln[P_i(X_F)/P_i(X_I)] \int_{\alpha=0}^1 [\tilde{S}_i(0) + \alpha(\tilde{S}_i(1) - \tilde{S}_i(0))] d\alpha \\ &= \exp \sum_i \ln[P_i(X_F)/P_i(X_I)] * \frac{(\tilde{S}_i(0) + \tilde{S}_i(1))}{2} = \Pi_i [P_i(X_F)/P_i(X_I)]^{(\tilde{S}_i(0) + \tilde{S}_i(1))/2} \\ &= [P(\text{Las}, X_F)/P(\text{Las}, X_I)]^{1/2} [P(\text{Paa}, X_F)/P(\text{Paa}, X_I)]^{1/2} . \end{aligned}$$

While the Divisia indexes (43.6) and (43.7) are usually interpretable as averages of Laspeyres and Paasche indexes, it is reasonable to ask whether they are too complicated. Perhaps we should have used Laspeyres or Paasche indexes or explicit combinations of the two. The advantage of Divisia indexes is that they have simple deviation forms [(43.1) and (43.2)]. For Laspeyres indexes, the deviation forms are

$$p = \sum_i S_i(0, X_I, X_F) p_i \quad \text{and} \quad q = \sum_i S_i(0, X_I, X_F) q_i ,$$

necessitating the retention of the initial shares, $S_i(0, X_I, X_F)$, in all steps of Johansen/Euler computations. With Paasche indexes, we need to implement in Johansen/Euler computations an iterative procedure involving guesses of the final shares, $S_i(1, X_I, X_F)$. Because it is the vehicle for all our computing and most of our thinking and interpreting, we were keen to keep the deviation version of our model as simple as possible. Thus, despite their complicated levels representations, we chose to use Divisia indexes.

44. Appendix: the TABLO implementation of the algorithm for handling forward-looking expectations of rates of return

We set out the technical aspects of the algorithm discussed in section 30 for computing MONASH solutions with forward-looking expectations. The algorithm is implemented mainly: by formulas in subsection 18.5t; by the updating of the coefficient EROR_G in subsection 18.7; by the calculation of $d_error(j)$ via E_d_f in subsection 18.8n; and by the arrangement of input and output files.

The arrangement of files is illustrated in Figure 44.1. In the figure we assume that there are 30 forecast iterations followed by a forecast rerun (iteration 31) and 5 policy iterations (iterations 32 to 36). The forecast iterations produce solutions for years 1 to 20, and the forecast rerun and policy iterations produce solutions for years t_a to 20 where t_a is the first year in which the policy shocks are anticipated. Policy solutions for years before t_a are not required as these would be identical to forecast solutions. The computation of the solution for year q in iteration n is denoted in the figure by $Sn.q$.

We assume that there are three data input files with logical names F, I and R. A minimum of three files is required. As indicated in the diagram the inputs to S2.1 are drawn from two processes: data from year zero (F0) and output from S1.20 (I1.20 and R1.20). Since the F input is from a different process to that generating the I and R inputs, the F file must be separate from I and R. The inputs to S31. t_a are also drawn from two processes: output (I30.20) from the previous solution, S30.20; and output [F30. (t_a-1) and R30. (t_a-1)] from an earlier solution, S30. (t_a-1) .⁸⁶ Since the I input is from a different process to that generating the F and R inputs, the I file must be separate from F and R. Hence all three files must be separate.

The input file F can be thought of as containing input-output data. The forecast iterations each start with input-output data for year zero. Throughout these iterations the input-output data are updated and passed from year to year. The forecast rerun and each of the policy iterations start with input-output data generated by the final forecast solution for year t_a-1 . As in the forecast iterations, these input-output data are updated and passed from year to year. In addition to input-output data the F file can contain substitution parameters and other data of no particular interest in the current context.

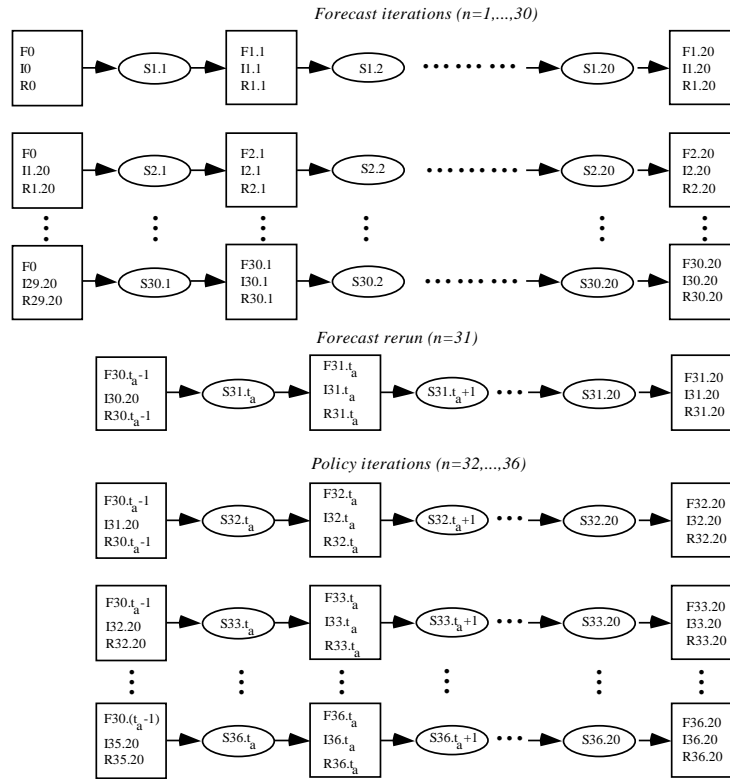
The input file I contains just one data item, ITER_NUM. Through its base value ITER_NUM_B, ITER_NUM informs each solution S $n.q$ of the iteration currently in progress. In subsection 18.5t, ITER_NUM_B affects the values of ONE_ITER1, ONE_IT1_REP, ZERO_PYR1 and DUM_IT1. As required by our algorithm, these coefficients allow expected rates of return to be determined differently in different iterations.

To see how ITER_NUM keeps track of the iteration number, we first need to look at the coefficients YEAR_B, YEAR and NYEARS. YEAR_B is the base value of the coefficient YEAR (see subsection 18.5t) which is read from the F file. In F0, YEAR has the value 1. During any solution, the value of YEAR is increased by 1 via updating by del_unity (subsection 18.7). Thus, in any data file F $n.(q-1)$ the value of YEAR is q giving YEAR_B a value of q in solution S $n.q$ for all n , that is in all solutions the value of YEAR_B denotes the year of the solution. NYEARS is a parameter giving the number of forecast years, 20 in Figure 44.1. Now we return to the valuation of ITER_NUM. In I0, its value is 1. As can be seen from subsection 18.7, ITER_NUM is updated in each solution (S $n.q$) by the value of ITER_ADJUST. This value is determined by a Formula(Initial) in subsection 18.5t. It is 0 except when YEAR_B = NYEARS, in which case it is 1. Thus, in the final year of each iteration ITER_NUM moves up by 1. Consequently, ITER_NUM has the value 1 in I0 to I1.19, the value 2 in I1.20 to I2.19, and the value 36 in

⁸⁶ If $t_a = 1$ then we define F $n.(t_a-1)$ as F0 and R $n.(t_a-1)$ as R $(n-1).20$.

I35.20 to I36.19. This means that ITER_NUM_B has the value n throughout the nth iteration.

Figure 44.1. Solving MONASH with Forward-looking Expectations



The main role of the third data file, R, is to carry values for the coefficients $EROR_G(j,q)$, where j is an industry ($j \in IND$) and q ($q \in TIME$) is the name of a year, $T0, T1, \dots, T20$. Values of $EROR_G$ are read at the beginning of each solution and then used, via a Formula(Initial) in subsection 18.5t, in setting values for the coefficients $EROR_G_B(j,q)$. Particular columns are picked out of $EROR_G_B$ to set values for $EROR_F(j)$ and $EROR_B(j)$. As explained in subsection 30.2, $EROR_B(j)$ and $EROR_F(j)$ are used via E_d_f in setting the movement in j 's expected rate of return [$d_eror(j)$].

From the formulas in subsection 18.5t, it is apparent that we need to look at the coefficients DUM_TIME , DUM_TIME_LAG and $COEFF_TIME$ to understand which columns of $EROR_G_B$ are picked out in setting $EROR_F$ and $EROR_B$.

COEFF_TIME is a vector of parameters (no updates) taking values for all q in the set TIME. These values, which are read initially from F0, are 0, 1, 2, ..., 20. From its formula in subsection 18.5t, we see that DUM_TIME(q) is 1 if COEFF_TIME(q) = YEAR_B, otherwise it is zero. Consequently, in year q computations, DUM_TIME is a vector with a 1 in the position corresponding to year q and 0 elsewhere. Similarly, in year q computations, DUM_TIME_LAG is a vector with a 1 in the position corresponding to year $q-1$ and 0 elsewhere.

Now we see from the formulas in subsection 18.5t that EROR_F in computations for year t is the column EROR_G_B(j,q), $q = Tt$. Similarly, EROR_B is the column EROR_G_B(j,q), $q = T(t-1)$.

As explained in subsection 30.2, EROR_F and EROR_B do not influence d_eror in either the first forecast or first policy iteration. However for other iterations we must ensure in the year t computation that EROR_G_B(j,Tt) has the value we wish to assume for j 's expected rate of return in year t and EROR_G_B($j,T(t-1)$) has the value we wish to assume for j 's expected rate of return in year $t-1$.

In the file R0 used in S1.1, EROR_G is a matrix of zeros. In all solutions (S1. q) in the first iteration, the coefficients ZERO_PYR1 and DUM_IT1 are 1. Thus, as can be seen from the Formula(Initial)s in subsection 18.5t for EROR_G_B and EROR_G, EROR_G is altered at the beginning of S1.1 so that its T0 column is replaced by the vector ROR_ACT_L_B. This is the vector of actual rates of return in year -1 and is computed by formulas in subsection 18.5t using data for years -1 and 0. In the course of S1.1, the T0 column of EROR_G is updated by changes in actual lagged rates of return ($d_ror_act_l$), see subsection 18.7. At the end of S1.1, the EROR_G matrix in R1.1 has actual rates of return for year 0 in its T0 column and zeros elsewhere.

The R1.1 file is taken into S1.2. As in S1.1, the EROR_G matrix is immediately altered via Formula(initial)s. This time the year 1 column is replaced by ROR_ACT_L_B which is now the vector of actual rates of return for year 0. These actual rates of return are calculated using data for year 0 and data for year 1 generated in S1.1. In the course of S1.2, the year 1 column of EROR_G is updated by changes in actual lagged rates of return ($d_ror_act_l$). At the end of S1.2, the EROR_G matrix in R1.2 has actual rates of return for year 0 in its T0 column, actual rates of return for year 1 in its T1 column and zeros elsewhere.

At the end of the first iteration the EROR_G matrix in R1.20 contains all the actual rates of return implied in the first iteration. More specifically, columns T0 to T19 contain the actual rates of return for years 0 to 19 implied by data for year zero⁸⁷ and by the sequence of solutions S1.1 to S1.20. The T20 column continues

⁸⁷ As is apparent from our description of alterations to EROR_G in S1.1, data for year -1 were used in the process of forming the T0 column. However these data do not influence the value of the T0 column emerging from S1.1 and thus appearing in R1.20.

to be a vector of zeros. However, as is clear from the discussion of E_d_f in subsection 30.2, the T20 column plays no role in the computations (COEFF_NYEAR is zero in the last year).

R1.20 is used as part of the input to S2.1. Because in all forecast iterations beyond the first, DUM_IT1 is 0, EROR_G_B in S2.1 is simply EROR_G from R1.20. Consequently, the values of EROR_B and EROR_F used in S2.1 as expected rates of return ($EROR_{jt}^2$, $t = 0, 1$) are the actual rates of return for years 0 and 1 implied in the first iteration: that is, as required by our algorithm [see (30.21)]

$$EROR_{jt}^2 = ROR_ACT_{jt}^1 \quad \text{for } t = 0, 1 \quad . \quad (44.1)$$

At the start of S2.1 (but after the evaluation of EROR_G_B), the T0 column of EROR_G is altered by a Formula(Initial) in subsection 18.5t so that it becomes $EROR_G(j, T0) =$

$$EROR_G_B(j, T0) + ADJ_RE(j) * [ROR_ACT_L_B(j) - EROR_G_B(j, T0)] ,$$

that is,

$$EROR_G(j, T0) = EROR_{j0}^2 + ADJ_RE(j) * [ROR_ACT_{j(-1)}^2 - EROR_{j0}^2] \quad . \quad (44.2)$$

During S2.1, the T0 column of EROR_G is updated (subsection 18.7) by $ADJ_RE(j) * d_ror_act_l(j)$. Consequently, by the end of S2.1, the T0 column of EROR_G in R2.1 is given by

$$EROR_G(j, T0) = EROR_{j0}^2 + ADJ_RE(j) * [ROR_ACT_{j0}^2 - EROR_{j0}^2] , \quad (44.3)$$

while the other columns remain unchanged from their values in the EROR_G matrix in R1.20.

R2.1 is used as part of the input to S2.2. Thus, EROR_B and EROR_F in S2.2 are the actual rates of return for years 1 and 2 implied in the first iteration: that is, as required by our algorithm [see (30.21)]

$$EROR_{jt}^2 = ROR_ACT_{jt}^1 \quad \text{for } t = 1, 2 \quad . \quad (44.4)$$

During S2.2, the T1 column of EROR_G is moved so that by the end of S2.2 it is given by

$$EROR_G(j, T1) = EROR_{j1}^2 + ADJ_RE(j) * [ROR_ACT_{j1}^2 - EROR_{j1}^2] . \quad (44.5)$$

On continuing to examine the evaluations of EROR_F and EROR_B, we see that the TABLO code for MONASH implemented with the file flow in Figure 44.1 satisfies (30.21) throughout S2.1 to S2.19: that is

$$EROR_{jt}^2 = ROR_ACT_{jt}^1 \quad \text{for all } t < 20 \quad . \quad (44.6)$$

In S2.20 we set $d_error(j) = 0$ (recall that COEFF_NYEAR = 0 in the final year). In light of (30.20), this ensures that

$$\text{EROR}_{jt}^2 = \text{ROR_ACT}_{jt}^1 \quad \text{for } t = 20 \quad . \quad (44.7)$$

On continuing to examine the updating of the EROR_G matrix, we see that by the end of iteration 2, the EROR_G matrix which forms part of R2.20 is given by

$$\text{EROR_G}(j, Tq) = \text{EROR}_{jq}^2 + \text{ADJ_RE}(j) * [\text{ROR_ACT}_{jq}^2 - \text{EROR}_{jq}^2] \quad \text{for all } q < 20, \quad (44.8)$$

and

$$\text{EROR_G}(j, T20) = 0 \quad . \quad (44.9)$$

In the solutions for the first 19 years of forecast iteration 3, the expected rates of return are given by values present in the EROR_G matrix generated in iteration 2. Thus, in accordance with (30.23a)

$$\text{EROR}_{jq}^3 = \text{EROR}_{jq}^2 + \text{ADJ_RE}(j) * [\text{ROR_ACT}_{jq}^2 - \text{EROR}_{jq}^2] \quad \text{for all } q < 20, \quad (44.10)$$

and with $d_eror(j)$ set at 0 in year 20, equation (30.23b) is also satisfied. The EROR_G matrix emerging from iteration 3 is given by

$$\text{EROR_G}(j, Tq) = \text{EROR}_{jq}^3 + \text{ADJ_RE}(j) * [\text{ROR_ACT}_{jq}^3 - \text{EROR}_{jq}^3] \quad \text{for all } q < 20, \quad (44.11)$$

and

$$\text{EROR_G}(j, T20) = 0 \quad . \quad (44.12)$$

This provides the values for the expected rates of return required in iteration 4.

After 30 forecast iterations, we perform the forecast rerun iteration ($n=31$). As explained in subsection 30.2, in this iteration we switch to the policy closure and shock the exogenous variables with their values from iteration 30. The F and R files used in the first year (t_a) of the forecast rerun are those used as inputs to year t_a in the last forecast iteration. Only the I file, carrying the iteration number, differentiates the inputs to $S31.t_a$ from those to $S30.t_a$. The difference between the data in $I30.20$ (the input to $S31.t_a$) and $I30.(t_a-1)$ (the input to $S30.t_a$) affects the values of ITER_NUM and ITER_NUM_B but has no effect on any other coefficient and therefore causes no differences between the solutions generated in iterations 31 and 30.⁸⁸

The F and R files used in the first year of the first policy iteration are those used as inputs to year t_a in both the last forecast iteration and in the forecast rerun. Only the I file ($I31.20$) used in $S32.t_a$ differs from the corresponding inputs to $S30.t_a$ and $S31.t_a$.

⁸⁸ ITER_NUM_B appears on the RHSs of the formulas in subsection 18.5t for ONE_ITER1 , ONE_IT1_REP , ZERO_PYR1 and DUM_IT1 . The values of these coefficients do not change when ITER_NUM_B moves from 30 to 31, that is from NOFITERS to $\text{NOFITERS}+1$.

With I31.20 as an input to S32.t_a, the value of ITER_NUM_B throughout the first policy iteration is given by

$$\text{ITER_NUM_B} = N + 2 \quad (44.13)$$

where N (=30) is the number for forecast iterations (denoted in the TABLO code by NOFITERS). Under (44.13), ONE_IT1_REP is set at 1. In combination with the settings of the other coefficients in E_d_f, this gives

$$d_error(j) = d_error_o(j) + del_ror_se(j) - del_ror_se_o(j) \text{ in S32.t for } t = t_a \text{ to } 19, \quad (44.14a)$$

and

$$d_error(j) = 0 \text{ in S32.t for } t = 20 \quad (44.14b)$$

where

$d_error(j)$ is the change in j 's expected rate of return between $t-1$ and t in the first policy iteration ($n = N+2 = 32$);

$d_error_o(j)$ is the change in j 's forecast expected rate of return between $t-1$ and t derived from the forecast rerun ($n = N+1 = 31$);

$del_ror_se(j)$ is the change in j 's rate of return under static expectations between $t-1$ and t in the first policy iteration ($n = N+2 = 32$); and

$del_ror_se_o(j)$ is the change in j 's rate of return under static expectations between $t-1$ and t derived from the forecast rerun ($n = N+1 = 31$).

Equations (44.14a & b) are change versions of (30.26a & b). Thus, to be sure that (30.26a & b) are satisfied as required by our algorithm, it is sufficient to show that

$$\text{EROR}_{jt}^{N+2} = \text{EROR}_{jt}^{N+1} + \text{ROR_SE}_{jt}^{N+2} - \text{ROR_SE}_{jt}^{N+1} \quad \text{for } t = t_a - 1. \quad (44.15)$$

This equation is satisfied because in year t_a-1 expected and static rates of return (ERORs and ROR_SEs) are the same in the first policy iteration ($n = N+2$) as in the forecast rerun ($n = N+1$). Both ROR_SE_{jt}^{N+2} and ROR_SE_{jt}^{N+1} , $t = t_a-1$, are calculated from data in F30.(t_a-1) and both EROR_{jt}^{N+2} and EROR_{jt}^{N+1} , $t = t_a-1$, are taken from the T(t_a-1) column of the EROR_G matrix in the same R file, R30.(t_a-1).⁸⁹

As in the first forecast iteration, in the first policy iteration the EROR_G matrix plays no role in the determination of the $d_error(j)$ s. These are determined by (44.14a & b). However, during the first policy iteration the EROR_G matrix is altered by Formula(Initial)s and updates, so that it can play a role in the determination of the $d_error(j)$ s in the second policy iteration. With one difference

⁸⁹ In the first year of any policy iteration, ZERO_PYR1 is 0 (see subsection 18.5t). Consequently, in S32. t_a the expected rates of return, EROR_B, assumed for year t_a-1 are those in the T(t_a-1) column of the EROR_G matrix in R30.(t_a-1). In the forecast rerun, DUM_IT1 is 0. Consequently, in S31. t_a the expected rates of return, EROR_B, assumed for year t_a-1 are also those in the T(t_a-1) column of the EROR_G matrix in R30.(t_a-1).

the alterations are carried out in the same way as those in the first forecast iteration. The difference is that whereas all the columns T0 to T19 of the EROR_G matrix were altered during the first forecast iteration, only the columns T_{t_a} to T19 are altered during the first policy iteration. At the end of the first policy iteration columns T_{t_a} to T19 in the EROR_G matrix in R32.20 contain actual rates of return implied in the first policy iteration. Columns T0 to T_{t_a-1} and T20 remain at their values in R30. (t_a-1) .

From here, the policy iterations proceed in the same way as the forecast iterations. In each policy iteration, expected rates of return are set via the EROR_G matrix to reflect actual rates of return in previous policy iterations, and the EROR_G matrix is updated to carry information on expected rates of return to subsequent iterations.